AN INVESTIGATION OF EFFECTS OF BOUNDARY LAYER THICKNESS ON A THIN FILM OF LIQUID FLOW DOWN AN INCLINED PLANE

BY

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DECLARATIONS

Declaration by the Candidate

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DEDICATION

I dedicate this work to my Brother Sailas Kipyegon, sisters Judith Chepkoech and Audrey Chepn'geno, cousins Mercy Chepng'etich, Ian Kiplang'at, Emmanuel Kibet and Niece Faith Chepkurui and Nephew Kiplang'at.

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ABSTRACT

The motion of fluid substances can be described by the Navier-Stokes equations. These equations arise from the application of Newton's second Law of motion to a fluid. In this study, Navier-Stokes equations in two dimensions have been taken into consideration. They are then applied to an incompressible viscous fluid motion down an inclined plane with net flow. These leads to finding the boundary layer thickness and examining the effects to the velocity of the motion at various angles of inclination. Flow of viscous laminar incompressible fluid does not always flow in horizontal position but sometimes on an inclined position. This makes it necessary to investigate the flow on an inclined plane. Most solutions that have been obtained are of the flow over horizontal flat plate. Solution that has been obtained for flow over a plane flat an angle of inclination was done by an experiment involving a at photographic film being pulled up by a processing bath by rollers at an angle θ to the horizontal but it is found that boundary layer thickness of the flow is obtained where there is no net flow and the angle of inclination is not varied to show the effects on the velocity of the flow. Linear and quadratic polynomials and sine function approximate velocity profiles have been obtained under initial boundary layer conditions. These velocity profiles have been used in momentum integral equation for flow over an inclined plane to get the boundary layer thickness. Boundary layer thickness is one of the parameters that is used to obtain the flow velocity down inclined plane.

NOTATIONS

∇ :	Gradient operator
μ :	dynamic viscosity
$\overline{V}$:	velocity vector
ν :	kinematic viscosity
$\frac{D}{Dt} = \frac{\partial}{\partial t} + v.\nabla$:	material derivative operator
u :	velocity component in x- direction
v :	velocity component in y- direction
w :	velocity component in z- direction
ρ :	density of fluid
ΔQ :	change in a Q quantity
θ :	angle of inclination
δ :	boundary layer thickness
b_x :	gravity in x- direction
b_y :	gravity in y- direction
b_z :	gravity in z- direction
m :	mass
$\dot{m} = \frac{dm}{dt}$:	change in mass per unit time
F :	force
σ :	normal stress
τ :	viscous shear stress

τ_w : viscous shear stress on the solid surface

U_∞: velocity vector of the main stream

P: pressure

M: momentum

V: volume

g: gravity vector

Re : Reynolds number

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CHAPTER ONE

1.0 INTRODUCTION

1.1 Background of the study

A fluid is a substance that continually deforms (flows) under an applied shear stress, no matter how small. Fluids are a subset of the phases of matter and include liquids, gases, plasmas and, to some extent, plastic solids. Contribution made by Prandtl to fluid motion clarified the essential influence of viscosity in flows at high Reynolds number Schlichting (1955). As a result of this influence of viscosity when fluid flows on surface of solid bodies, the fluid seems to "stick" to the surface. These viscous forces originate from molecular interactions. Right at the surface the flow has zero relative speed and the fluid transfers momentum to adjacent layers through the action of viscosity. Thus a thin layer (boundary layer) of fluid with lower velocity than the outer flow develops. The requirement that the flow at the surface has no relative motion is the \no slip condition." The thickness of the velocity boundary layer is normally defined as the distance from the solid body at which the flow velocity is 99 per cent of the freestream velocity, that is, the velocity calculated at the surface of the body in an inviscid flow solution. The no-slip condition requires the flow velocity at the surface of a solid object be zero and the fluid temperature be equal to the temperature of the surface. The flow velocity will then increase rapidly within the boundary layer, governed by the boundary layer equations. Navier-Stokes equations commonly abbreviated N-S equations are some of the most important equations that describe the physics of a large number of phenomena that occur both in academic and economic interests. These equations may be used to model a range of phenomena namely weather, ocean currents, water flow in a pipes and flow around an aerofoil (wing) etc. These equations can be used in the design of aircrafts and cars, the study of blood flow, the design of hydroelectric power stations, the analysis of the effects of pollution, etc. Coupled with Maxwell's equations N-S equations can be used to model and study magnetohydrodynamics(MHD).

Study of effects of viscosity, in two dimensional (2-D) flows and the theory of boundary layer, utilizes Navier-stokes equations too. In most cases when working out the solution of N-S equations of fluid flows over a surface of a at plane, force of gravity is neglected, but we have to use force of gravity when studying flows of fluid down an inclined plane because the motion is influenced by force of gravity.

These are definitions and explanations to terms and concepts that will be used commonly.

Definition 1.1. Steady and unsteady flows.

A fluid flow is said to be steady if the fluid properties do not change with time, otherwise the fluid flow is unsteady.

Definition 1.2. Laminar and turbulent flow.

A fluid flow is said to be laminar if the fluid particles move along well defined streamlines or along paths that are straight and parallel otherwise the fluid flow is said to be turbulent.

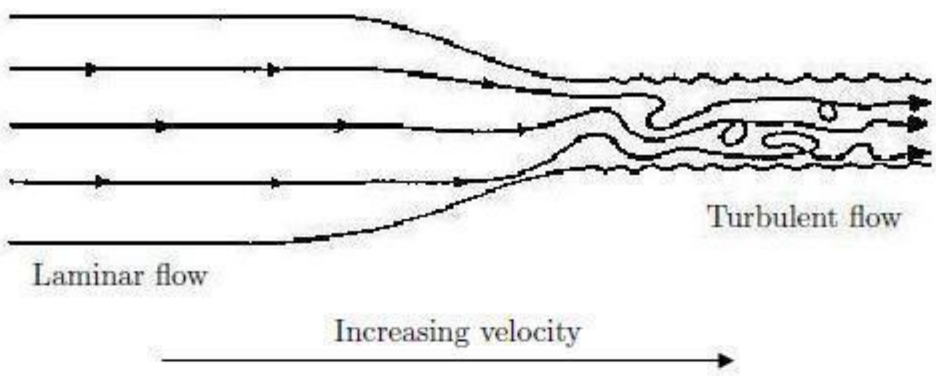

Figure 1.1: Laminar and turbulent flow

Definition 1.4 Newtonian and Non-Newtonian fluid.

Given two parallel plates suspended in a liquid, and separated by a small distance y in y-axis, if the upper plane is kept stationary while the lower plate is set to motion with a velocity u then the Newton's law of viscosity states that the shear stress between adjacent fluid layers is proportional to the negative value of the velocity gradient between the two layers. That is

$$\text{shear stress} = \mu \frac{du}{dy} \tag{1.1.1}$$

Definition 1.4 Newtonian and Non-Newtonian fluid.

Newtonian fluid is a fluid that obeys Newton's law of viscosity otherwise is considered Non-Newtonian fluid.

Definition 1.5 Incompressible and compressible flow.

A fluid flow is said to be incompressible if the fluid density ρ does not change from point to point with time in the flow, otherwise it is considered to be compressible.

Definition 1.6 Viscous flows.

Viscous fluid flows are those in which fluid friction has significant effects on the fluid motion.

1.1.1 Conservation of mass

The principle of conservation of mass states that, except in nuclear process Conservation law (2012), matter is neither created nor destroyed. Given an infinitesimal volume about a point e.g $p(x, y, z)$ considered in a fluid medium where $u, v,$ u, and w are components of fluid velocity at p, in the directions of $x-$, $y-$, and $z-$ axes respectively, relative to the chosen volume, the principle of conservation of mass is given by:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \bar{V}) = 0, \tag{1.1.2}$$

where ρ is the density and v is the velocity of the fluid. Equation (1.1.2) above is known as continuity equation and if the flow is steady, then $\frac{\partial \rho}{\partial t} = 0$ and the continuity equation reduces to

$$\nabla.(\rho \bar{V}) = 0 \tag{1.1.3}$$

1.1.2 Conservation of linear momentum

The principle of conservation of momentum states that the time rate of change of linear momentum within the system equals the net transport rate of linear momentum into the system by external forces and by mass flow Richards (2008). The following expressions of linear momentum for a differential control volume in 2-D flow;

x-component; $$\frac{\partial(\rho u)}{\partial t} = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial _{yx}}{\partial y} + \rho b_x - \left[\frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho vu}{\partial y}\right] \tag{1.1.4}$$

y-component; $$\frac{\partial(\rho v)}{\partial t} = -\frac{\partial P}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \rho b_y - \left[\frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho vv}{\partial y}\right] \tag{1.1.5}$$

where b_x and b_y are $x-$ and $y-$ components of gravity respectively in equations, τ_{xx} is the viscous shear stress in the x-face, x- direction , τ_{yx} is the viscous shear stress in the $y-$face, x- direction, τ_{yy} is the viscous shear stress in the y-face, y-direction, τ_{xy} is the viscous shear stress in the x-face, y- direction and P is the pressure.

1.1.3 Navier-Stokes equations

Consider the continuity equation for steady flow, and linear momentum for a differential control volume in 2-D flow. Note that N-S equations are not a combination of continuity and momentum equations. Considering equation (1.1.3), we have Navier-Stokes equations as;

$$\frac{DV}{Dt} = -\frac{1}{\rho}\nabla P + g + \nu\nabla^2\overline{V} \qquad (1.1.6)$$

1.1.4 Momentum integral equation

Navier-Stokes equations together with the assumptions that the fluid flow is viscous laminar, steady and incompressible will lead to finding the boundary layer thickness δ, by the method of momentum integral approach Mei (2002). Momentum integral equation is given by

$$\frac{d}{dx}\int_0^\delta \left(uU_\infty - u^2\right)dy + \frac{dU_\infty}{dx}\int_0^\delta \left(U_\infty - u\right)dy = \frac{\tau_w}{\rho} \qquad (1.1.6)$$

Where U_∞ is the main stream velocity.

1.1.5 Inclined plane

This is a plate tilted at an angle θ to the horizontal surface as shown in Figure 1.2.

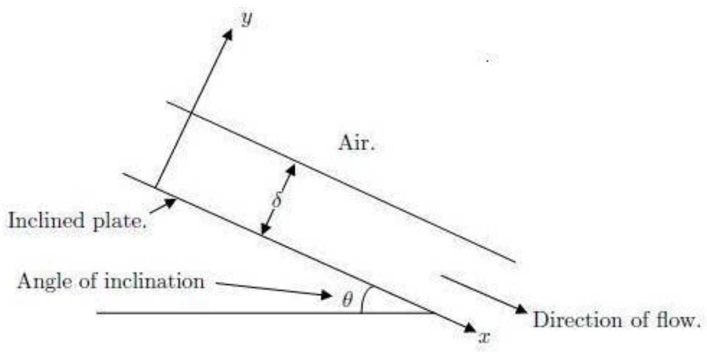

Figure 1.2: Inclined plane

1.2 Derivation of Navier Stokes Equations

1.2.1 Conservation of Mass

Except in nuclear Process, matter is neither created nor destroyed Conservation law (2012). This is the principle of conservation of mass to a flowing fluid.

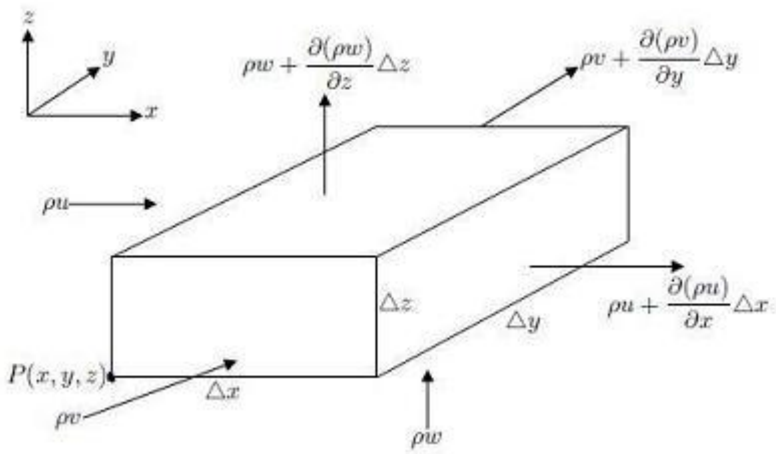

Figure 1.3: Mass Flux

In the figure 1.3 above, an infinitesimal volume about a point $P(x, y, z)$ is considered in a fluid medium u, v and w are components of fluid velocity at P, in the $x-$, $y-$, and $z-$ axes respectively, relative to the chosen volume.

The mass entering per unit time and area of the left face in the $x-$direction is; ρu and corresponding mass flux leaving the right face is;

$$\rho u + \frac{\partial(\rho u)}{\partial x}\Delta x .$$

The increment is written by the use of Taylor series. The area of either face in the $x-$direction is $(\Delta y \Delta z)$. Considering all the three directions, it is noted that the net mass leaving the volume per unit time is;

$$\left[\frac{\partial(\rho u)}{\partial x}\Delta x\right]\Delta y.\Delta z \qquad \text{in the } x-\text{direction,}$$

$$\left[\frac{\partial(\rho v)}{\partial y}\Delta y\right]\Delta x.\Delta z \qquad \text{in the } y-\text{direction,} \qquad (1.2.1)$$

$$\left[\frac{\partial(\rho w)}{\partial z}\Delta z\right]\Delta x.\Delta y \qquad \text{in the } x-\text{direction.}$$

The effect of mass loss in equation (1.2.1) is to cause the time rate of decrease of the mass encompassed by the volume : $m = \rho(dvol)$. Since $dvol = \Delta x \Delta y \Delta z$ is chosen not to change with the time we obtain the conservation of mass as:

$$\left[\frac{\partial(\rho u)}{\partial x}\Delta x\right]\Delta y.\Delta z + \left[\frac{\partial(\rho v)}{\partial y}\Delta y\right]\Delta x.\Delta z + \left[\frac{\partial(\rho w)}{\partial z}\Delta z\right]\Delta x.\Delta y = -\frac{\partial \rho}{\partial t}\Delta x \Delta y \Delta z$$

i.e

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = -\frac{\partial \rho}{\partial t}$$

(1.2.2)

Equation (1.2.2) can be re-written in form of equation (1.1.2) Robert et al.,(2005).

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \overline{V}) = 0 \qquad (1.2.3)$$

1.2.2 Conservation of Linear Momentum for a Differential Control Volume

When the rate-form of the conservation of mass equation to a differential control volume (open system) in Cartesian coordinates is applied, what is obtained is the continuity equation:

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] \tag{1.2.4}$$

The same thing is done for conservation of linear momentum. As with the development of the continuity equation, first apply the rate-form of conservation of linear momentum to a finite-size open system (a control volume) and then consider what happens in the limit as the size of the control volume approaches zero.

1.2.3 Conservation of Linear Momentum for a 2-D Flow

For simplicity, consider a two-dimensional, unsteady flow subject to a body force, b. The body force, the density, and the velocity all are field variables and may vary with position (x, y) and time.

In words, the time rate of change of the linear momentum within the system equals the net transport rate of linear momentum into the system by external forces and by mass flow. To apply this principle, select a small finite-size volume in the flow field, $\Delta x \Delta y \Delta z$, where Δz is the depth of the small volume, i.e. in the direction of the $x-$axis. Because of the multiple momentum transfers in a moving fluid, focus on only one component of linear momentum at a time.

Begin with the $x-$component of linear momentum, referred to hereafter as the "$x-$Momentum". To identify the transfers of $x-$momentum, enlarge the small volume and show all transfers of $x-$momentum on the diagram of the control volume (see Figure 1.4). Linear momentum can be transported into the system by surface forces acting on all four sides, by body forces within the system, and by mass flows across all four sides of the control volume.

Writing the rate-form of the conservation of linear momentum equation for the $x-$momentum gives:

$$\frac{dM_x}{dt} = F_{x,net|surface} + F_{x,net|body} + (\dot{m}_x u)_{net} + (\dot{m}_y u)_{net} \tag{1.2.5}$$

The left-hand side of equation (1.2.5) represents the rate of change of $x-$momentum inside the system.

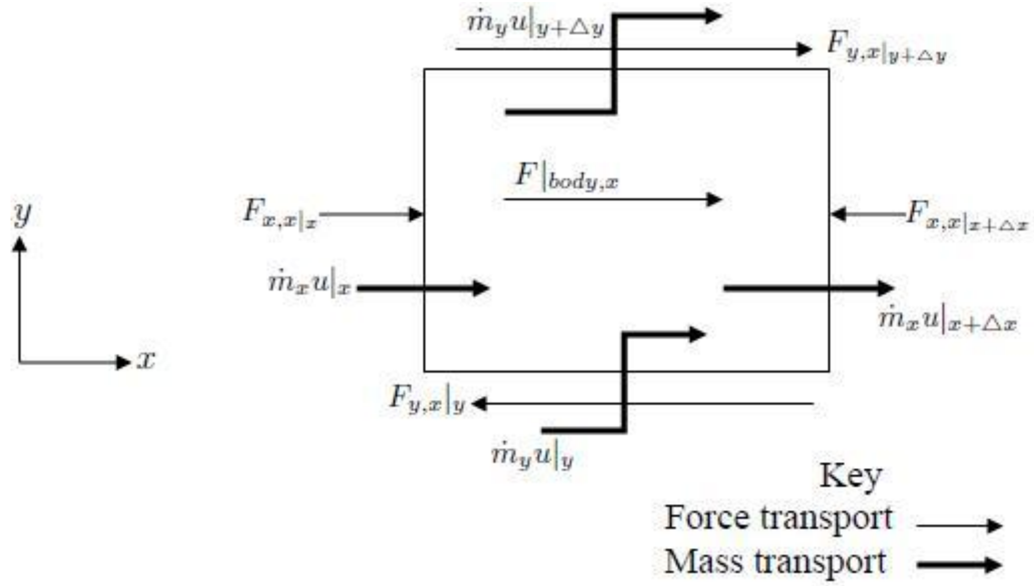

Figure 1.4: Transport rates of the x-component of linear momentum for the differential control volume.

The terms on the right-hand side of equation (1.2.5) correspond to the transport rates of $x-$momentum illustrated in Figure 1.4.

1.2.4 Rate of change of x-momentum inside the control volume

The $x-$momentum inside the control volume can be written as follows using the meanvalue theorem from calculus:

$$M_{x,sys} = \int_{\Delta x \Delta y \Delta x} u\rho dV = \overline{\rho u}\Delta x \Delta y \Delta z \tag{1.2.6}$$

where the bar notation $\overline{\rho u}$ indicates an average or mean value. The mean-value theorem says that the integral of a continuous function ρu over a volume equals the product of the volume and the mean value of the continuous function within the volume; thus, a simple product can replace the integral.

To find the rate-of-change of the $x-$momentum inside the control volume, the left hand side of Eq. (1.2.5), we evaluate the derivative

$$\frac{dM_x}{dt} = \frac{d}{dt}\left(\overline{\rho u}\Delta x \Delta y \Delta z\right) = \Delta x \Delta y \Delta z \frac{\partial(\overline{\rho u})}{\partial t} \tag{1.2.7}$$

Because we are holding x, y, and z constant during our differentiation the result is a partial derivative.

1.2.5 Net transport rate of x-momentum by external forces

The net transport rate of $x-$momentum by body forces is related to b_x, the component of the body force acting in the positive $x-$direction, and can be written as

$$F_{x,net|body} = \int_{\Delta x \Delta y \Delta z} b_x \rho dV = \overline{\rho b_x}\left(\Delta x \Delta y \Delta z\right) \tag{1.2.8}$$

The net transport rate of $x-$momentum by surface forces is related to external forces acting on each surface of the control volume. For a two-dimensional flow, the surface force for any surface can be decomposed into two components, a normal force and a shear force. Each force, shear or normal, is defined as the integral of the appropriate surface stress over the surface area.

The $x-$component of the surface force acting on the surface at $x + \Delta x$, is written as

$$F_{x,x|x+\Delta x} = \int_{\Delta y \Delta z} -\sigma_{xx} dA = -\left[\overline{\sigma_{xx}}\right]_{x+\Delta x} \Delta y \Delta z \tag{1.2.9}$$

where the minus sign occurs because the normal stress σ_{xx} by convention points out from the surface.

The x-component of the surface force acting on the surface at x, is written as

$$F_{x,x|x} = \int_{\Delta y \Delta z} -\sigma_{xx} dA = -\left[\overline{\sigma_{xx}}\right]_x \Delta y \Delta z \tag{1.2.10}$$

The $x-$component of the surface force acting on the surface at $y+\Delta y$, is written as

$$F_{y,x|y+\Delta y} = \int_{\Delta x \Delta z} \sigma_{yx} dA = \left[\overline{\sigma_{yx}}\right]_{y+\Delta y} \Delta x \Delta z \tag{1.2.11}$$

where the shear stress σ_{yx} points in the positive $x-$direction on a surface whose normal vector points in the positive $y-$direction.

The $x-$component of the surface force acting on the surface at y, is written as

$$F_{y,x}\Big|_y = \int_{\Delta x \Delta z} -\sigma_{yx} dA = -[\overline{\sigma_{yx}}]_y \Delta x \Delta z \tag{1.2.12}$$

where again the minus sign comes from the sign convention on the shear stress σ_{yx}.

Combining these forces to find the component of the net surface force acting on the control volume gives

$$F_{x,net|surface} = F_{x,x}\Big|_x - F_{x,x}\Big|_{x+\Delta x} + F_{y,x}\Big|_{y+\Delta y} - F_{y,x}\Big|_y$$

$$F_{x,net}\Big|_{surface} = \left[-\overline{\sigma_{xx}}\right]_x \Delta y \Delta z - \left[-\overline{\sigma_{xx}}\right]_{x+\Delta x} \Delta y \Delta z + \left[\overline{\sigma_{yx}}\right]_{y+\Delta y} \Delta x \Delta z - \left[\overline{\sigma_{yx}}\right]_y \Delta x \Delta z$$

$$F_{x,net}\Big|_{surface} = \left[\overline{\sigma_{xx}}\Big|_{x+\Delta x} - \overline{\sigma_{xx}}\Big|_x\right](\Delta y \Delta z) + \left[\overline{\sigma_{yx}}\Big|_{y+\Delta y} - \overline{\sigma_{yx}}\Big|_y\right](\Delta x \Delta z)$$

$$F_{x,net}\Big|_{surface} = \left[\frac{\partial \overline{\sigma_{xx}}}{\partial x} \Delta x\right](\Delta y \Delta z) + \left[\frac{\partial \overline{\sigma_{yx}}}{\partial y} \Delta y\right](\Delta x \Delta z)$$

$$F_{x,net}\Big|_{surface} = \left[\frac{\partial \overline{\sigma_{xx}}}{\partial x} - \frac{\partial \overline{\sigma_{yx}}}{\partial y}\right] \Delta x \Delta y \Delta z \tag{1.2.13}$$

1.2.6 Net transport rate of $x-$ momentum by mass flow

Finally, examine the net transport rate of $x-$momentum by mass flow. Mass flow occurs on all four surfaces of the control volume. To begin, determine the x momentum transport rate of $x-$momentum into the system with mass flow at x and $x+\Delta x$:

$$\left(\dot{m}_x u\right)_{net} = \left(\dot{m}_x u\right)_x - \left(\dot{m}_x u\right)_{x+\Delta x}$$

$$\left(\dot{m}_x u\right)_{net} = \left[\left(\rho u \Delta z \Delta y\right)_{in} u\right]_x - \left[\left(\rho u \Delta z \Delta y\right)_{out} u\right]_{x+\Delta x}$$

$$\left(\dot{m}_x u\right)_{net} = \left(\Delta z \Delta y\right)\left[\left(\rho u u\right)_x - \left(\rho u u\right)_{x+\Delta x}\right]$$

$$\left(\dot{m}_x u\right)_{net} = -\left(\Delta z \Delta y\right)\left[\frac{\partial(\rho u u)}{\partial x}\Delta x\right] \tag{1.2.14}$$

Now determine the transport rate of $x-$momentum into the system with mass flow at y and $y+\Delta y$:

$$\left(\dot{m}_y u\right)_{net} = \left(\dot{m}_y u\right)_y - \left(\dot{m}_x u\right)_{y+\Delta y}$$

$$\left(\dot{m}_y u\right)_{net} = \left[\left(\rho v \Delta z \Delta x\right)_{in} u\right]_y - \left[\left(\rho v \Delta z \Delta x\right)_{out} u\right]_{y+\Delta y}$$

$$\left(\dot{m}_y u\right)_{net} = \left(\Delta z \Delta x\right)\left[\left(\rho v u\right)_y - \left(\rho v u\right)_{y+\Delta y}\right]$$

$$\left(\dot{m}_y u\right)_{net} = -\left(\Delta z \Delta x\right)\left[\frac{\partial(\rho v u)}{\partial y}\Delta y\right] \tag{1.2.15}$$

Now that the expressions for $x-$component of the net surface force acting on the control volume, body forces, and each $x-$momentum transport or storage rate have been developed, i.e equations (1.2.13), (1.2.8) , (1.2.14), and (1.2.15), substitute these values back into equation (1.2.5).

Making the substitutions for each term we have the following expression: $\Delta x\Delta y\Delta z\dfrac{\partial(\overline{\rho u})}{\partial t}=\left[\dfrac{\partial\overline{\sigma_{xx}}}{\partial x}-\dfrac{\partial\overline{\sigma_{yx}}}{\partial y}\right]\Delta x\Delta y\Delta z+\overline{\rho b_x}(\Delta x\Delta y\Delta z)+$

$$\left[-\frac{\partial(\rho uu)}{\partial x}(\Delta x\Delta y\Delta z)\right]+\left[-\frac{\partial(\rho vu)}{\partial y}(\Delta x\Delta y\Delta z)\right] \qquad (1.2.16)$$

Dividing through by the volume $\Delta x\Delta y\Delta z$ and taking the limit as $\Delta x\to 0,\ \Delta y\to 0,$, and $\Delta z\to 0$, the average terms indicated by the bar notation $(\bar{\ })$ approach the value at the point x, y, t.

This gives the conservation of $x-$momentum for a differential control volume (two dimensional flow):

$$\frac{\partial(\rho u)}{\partial t}=\frac{\partial(\sigma_{xx})}{\partial x}+\frac{\partial(\sigma_{yx})}{\partial y}+\rho b_x-\left[\frac{\partial(\rho uu)}{\partial x}+\frac{\partial(\rho vu)}{\partial y}\right] \qquad (1.2.17)$$

A similar expression can be developed for the conservation of $y-$momentum for a differential control volume (two-dimensional flow):

$$\frac{\partial(\rho v)}{\partial t}=\frac{\partial(\sigma_{yx})}{\partial x}+\frac{\partial(\sigma_{yy})}{\partial y}+\rho b_y-\left[\frac{\partial(\rho vu)}{\partial x}+\frac{\partial(\rho vv)}{\partial y}\right] \qquad (1.2.17)$$

1.2.7 Revisiting the surface forces

Before proceeding, re-examine the surface forces. Surface forces include both pressure forces and viscous forces. The pressure force is a normal force produced by the local pressure p acting over a surface. The local pressure is an intensive property of the substance and is a normal stress whose value is independent of orientation Childress (2009). Viscous forces are produced by viscous stresses τ_{ij} that depend on fluid viscosity and velocity gradients in the flow. Viscous stresses strongly depend on orientation and for any surface can be decomposed into both normal stresses and shear stresses.

For a two-dimensional flow the surface stress σ_{ij} can be separated into a pressure term p and a viscous stress term τ_{ij} as shown below:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{yx} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix}$$

$$\sigma_{ij} = \begin{pmatrix} -P+\tau_{xx} & \tau_{yx} \\ \tau_{xy} & -P+\tau_{yy} \end{pmatrix} = \begin{pmatrix} -P & 0 \\ 0 & -P \end{pmatrix} + \begin{pmatrix} \tau_{xx} & \tau_{yx} \\ \tau_{xy} & \tau_{yy} \end{pmatrix}$$

Now the surface stress terms can be rewritten to clearly separate the pressure and viscous stresses. For example in equation (1.2.17), the surface stress terms can be rewritten as follows:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = \frac{\partial}{\partial x}\left(-P+\tau_{xx}\right) + \frac{\partial \tau_{yx}}{\partial y} = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} \tag{1.2.19}$$

Substituting the appropriate value for each stress back into Equations (1.2.17) and (1.2.18) and rearranging terms gives the following expressions conservation of linear momentum for a differential control volume 2-D flow:

$$x-\text{component;}\quad \frac{\partial(\rho u)}{\partial t} = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \rho b_x - \left[\frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho vu)}{\partial y}\right]$$

$$y-\text{component;}\quad \frac{\partial(\rho v)}{\partial t} = -\frac{\partial P}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \rho b_y - \left[\frac{\partial(\rho vu)}{\partial x} + \frac{\partial(\rho vv)}{\partial y}\right] \tag{1.2.20}$$

Note that there is now a clear distinction between the pressure terms and the viscous stress terms. To evaluate the viscous stresses we need to have a constitutive model for the fluid that describes how the shear stresses are related to the velocity gradients in the flow and to the fluid property known as viscosity. When a flow is inviscid, it has no viscosity and the viscous stress terms disappear. This greatly simplifies the mathematics and can give useful results under some conditions.

1.2.8 Navier-Stokes Equations for Incompressible 2-D Flow

Many important flows are essentially incompressible and this leads to significant simplifications. Additionally, restrict to Newtonian fluids. With these assumptions, we reproduce the Navier-Stokes equations for incompressible, two-dimensional flow. To develop these equations, first assume that the flow is incompressible and consider the consequences. The continuity equation reduces, Equation. (1.2.4) becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (1.2.21)$$

In Equation. (1.2.20), if flow is incompressible, then:

$$x-\text{component;} \quad \rho\frac{\partial u}{\partial t}=-\frac{\partial P}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\rho b_x-\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)$$

$$y-\text{component;} \quad \rho\frac{\partial v}{\partial t}=-\frac{\partial P}{\partial y}+\frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\rho b_y-\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) \qquad (1.2.24)$$

A Newtonian fluid is a fluid in which viscous stresses are proportional to the rate of angular deformation within the fluid Richards (2008). The constant of proportionality is $\mu -$ the dynamic viscosity. The viscous stresses for an incompressible, two-dimensional flow of a Newtonian fluid become Manyonge (2010):

$$\begin{aligned}\tau_{xx}&=2\mu\frac{\partial u}{\partial y},\\ \tau_{yy}&=2\mu\frac{\partial x}{\partial x}\\ \tau_{xy}&=\tau_{yx}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\end{aligned} \qquad (1.2.25)$$

Using these results, the viscous stress terms in equation (1.2.24) are replaced by terms containing the fluid viscosity and velocity gradients. Note that viscous stresses depend on both the fluid and the flow.

The next step is to substitute the viscous stress terms in equation (1.2.25) back into equation (1.2.24) and use the incompressible continuity equation for steady flow, equation (1.2.21), to simplify the expressions. Once done, recover the Navier-Stokes equations. The x-component of the Navier-Stokes Equation for a two-dimensional, incompressible flow is shown below:

$$\rho\frac{\partial u}{\partial t}=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\rho b_x-\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) \qquad (1.2.26)$$

This arrangement shows a clear connection to original conservation of linear momentum equation. When arranged in this form it results to Navier-Stokes Equations for a 2-D, Incompressible Flow Richards (2008),

$$x-\text{component};\ \rho\frac{\partial u}{\partial t}+\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial P}{\partial x}+\rho b_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$y-\text{component};\ \rho\frac{\partial u}{\partial t}+\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial P}{\partial y}+\rho b_y+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \qquad (1.2.27)$$

Following a similar process for three-dimensions, we can derive the full set of Navier Stokes Equations for an Incompressible Flow. These are stated below without explanation or development Richards (2008):

$$\begin{aligned}
x-;\quad & \rho\frac{\partial u}{\partial t}+\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial P}{\partial x}+\rho b_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\\
y-;\quad & \rho\frac{\partial v}{\partial t}+\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial P}{\partial y}+\rho b_y+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \qquad (1.2.28)\\
z-;\quad & \rho\frac{\partial w}{\partial t}+\rho\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial P}{\partial z}+\rho b_z+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)
\end{aligned}$$

The equations (1.2.28) stated above can be re-written as equation (1.1.6).

1.3 Statement of the problem

The boundary layer thickness approximation of a flow down an inclined plane with net flow using the momentum integral approach has not been done or presented by any author. This is tackled in this study. The velocity profiles approximations u_1, u_2, u_3 are to be obtained which are required to find the boundary layer thickness δ, of flow with net flow, then use it to find the effect of velocity by varying angles of inclination between $0 < \theta < \frac{\pi}{2}$

1.4 Objective of the study

1. To estimate the value of boundary layer thickness δ of the fluid flow with net flow.
2. To obtain the velocity of the fluid flow at different angles of inclination between $0 < \theta < \frac{\pi}{2}$.
3. To obtain the expression for the velocity u, down inclined plane.

1.5 Significance of the study

This study will be of great benefit in academics. It will also help soil engineers together with applied mathematicians in study of overland flows within small agricultural watersheds where the flow direction is controlled by the topography within the fields. This study also is not limited to the benefit in industrial coating processes.

CHAPTER TWO

2.0 LITERATURE REVIEW

Navier-Stokes equations were originally derived in the 1840s on the basis of conservation laws and first-order approximations. But if one assumed sufficient randomness in microscopic molecular processes they could also be derived from molecular dynamics, as done in the early 1900s Wolfram (2002).

The aerodynamic boundary layer was first defined by Ludwig Prandtl in a paper presented on August 12, 1904 at the third International Congress of Mathematicians in Heidelberg Robert et al., (2005), Germany. It allows aerodynamicists to simplify the equations of fluid flow by dividing the flow field into two areas: one inside the boundary layer, where viscosity is dominant and the majority of the drag experienced by a body immersed in a fluid is created and one outside the boundary layer where viscosity can be neglected without significant effects on the solution. This allows a closed-form solution for the flow in both areas, which is a significant simplification over the solution of the full Navier-Stokes equations. The majority of the heat transfer to and from a body also takes place within the boundary layer, again allowing the equations to be simplified in the flow field outside the boundary layer.

Mohanty (1994), wrote on laminar boundary layer of flow over a at plate that divides the flow of a real fluid past a solid body into two zones: a viscous layer surrounding a solid surface and a zero shear stress beyond it. Solution of boundary layer provides the methods of estimating the frictional resistance along the wetted surface of a body. He points out that the differential solutions are, however, not attainable without considering mathematical elegance and worked out the solution of momentum integral equation for flow over a flat plate.

Mei (2002), wrote in his lecture notes on fluid dynamics that with a general pressure gradient, the boundary layer equations can be solved by a variety of modern numerical means like finite element method. An alternative which can still be employed to simplify calculations is the momentum integral method of Karman Manyonge (2010). He explains this method for a transient boundary layer along the x-axis forced by an unsteady pressure gradient outside. This pressure gradient could be due to some unsteady and nonuniform flow such as waves or gust (sudden strong rush of wind).

Measurements to test the theory of boundary layer were carried out first by Burgers (1925), and particularly carefully and comprehensive measurements were reported later. It was found that the formation of boundary layer is greatly influenced by the shape of the leading edge as well as by the very small pressure gradient which may exist in the external flow. Corrections were introduced carefully for these possible effects, when he carried out his measurements on a plate in a stream of air.

Manyonge (2010), gave the derivation of Navier-stoke's equations in general form which applies to both compressible and incompressible flows. He also discusses on boundary layer theory where he gave two types of boundary layers i.e laminar boundary layer and turbulent boundary layer. In Laminar flow, Manyonge came up with the solution of N-S equations using Karman Momentum Integral equation approach for the flow over a horizontal flat plate. Sonin (2001), derived Navier-Stokes Equations of horizontal flows but could not derive flows on an inclined position where it is influenced by force of gravity.

According to a coating experiment involving a flat photographic film, being pulled up from a processing bath by rollers with a steady velocity U_∞ at an angle θ to the horizontal, Solution of Viscous-Flow Problems (2011). As the film leaves the bath, it entrains some liquid. The velocities at different angles are not given in order to find the thickness of the liquid that is required. However, the thickness is determined by a steady velocity moving up without net flows and not downwards with net flows. The expression for velocity (u) of viscous flow down an inclined plane was obtained by integration of N-S equations given initial and boundary conditions Nishikant (2011).

Numerical study of a thin liquid film owing down an inclined wavy plane has been done where the stability of a thin liquid film owing down an inclined wavy plane using a direct numerical solver based on a finite element/arbitrary Lagrangian Eulerian approximation of the free-surface NavierStokes equations. In this work Ern., (2011), the Nusselt flow which is a boundary layer type of flow featuring constant height, parabolic velocity profile, while the flow rate is determined by balancing the work of gravity with viscous dissipation but not at any height of boundary layer thickness.

Nishikant (2011), discussed on a viscous liquid that can flow while in contact with only one solid surface, the fluid motion being caused by a component of the gravity force parallel to the solid surface, obtaining the expression for velocity u but could not obtain the boundary layer thickness.

Grand et al. (2005), studied on Shape and motion of drops sliding down an inclined plane. The questions to be answered here were; what happens when a liquid drop slides down a uniform, inclined plane in a situation of partial wetting? At which velocity does it slide and what shape does it assume to accommodate capillary effects and drop motion? Here Le Grand did not find the boundary layer thickness.

CHAPTER THREE

3.0 FLOW DOWN AN INCLINED PLANE

Consider the motion of fluid which is caused by a component of a gravity force parallel to the inclined non-porous solid plane surface owing downwards, where the following assumptions are made; the fluid is Newtonian, laminar, and incompressible. Consider continuity equation (1.1.3) and momentum equation (1.1.6) which describes the steady flow.

Since the flow is steady, there is no change with time and in the flow direction. Therefore, the terms $\rho\frac{\partial u}{\partial t}, \rho\frac{\partial v}{\partial t}$, and $\rho\frac{\partial w}{\partial t}$ in equations (1.2.28) are equal to zero. Flow is not bounded in the z direction and therefore nothing happens in the z direction. Therefore,

$u = f(y)$ and $\frac{\partial u}{\partial x} = 0$ in $x-$direction,

$v = f(y)$ and $\frac{\partial v}{\partial y} = 0$ in $y-$direction and

$w = 0$ and $\frac{\partial w}{\partial z} = 0$ in $z-$direction.

Since the flow is in one direction, that is $x-$direction, then $v = 0$ and $w = 0$. This makes the terms

$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)$, $\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial u}{\partial z}\right)$, and $\rho\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)$ in equations (1.2.28) be equal to zero.

The Navier stokes equations (1.2.28) then reduces to;

$$x-\text{component;}\quad -\frac{\partial P}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2}+\rho b_x = 0 \qquad (3.0.1)$$

$y-$component; $-\dfrac{\partial P}{\partial y}+\rho b_y=0$ (3.0.2)

$z-$component; $-\dfrac{\partial P}{\partial z}+\rho b_z=0$ (3.0.3)

Since P is not a function of z, $\dfrac{\partial P}{\partial z}=0.$

Consider a viscous liquid owing while in contact with only one solid surface, the fluid motion is being caused by a component of the gravity force parallel to the solid surface. Such a plane flow is illustrated in Figure 1, showing a plane surface inclined above the horizontal by an angle θ and covered with a liquid layer of constant thickness δ that flows parallel to the plane in the downhill direction. The upper surface of the fluid $(y=\delta)$ is in contact with the air, in which the pressure is constant ($P=P_a$ at $y=\delta$, where P_a is the atmospheric pressure) and which exerts a negligible shear stress on the liquid surface ($\tau_{xy}=0$ at $y=\delta$).

To describe this motion, select the x component of the Navier-Stokes, noting that $\dfrac{Du}{Dt}=\dfrac{\partial u}{\partial t}+(V.\nabla)u=0$ and $(V.\nabla)u\neq 0$ since it is not a function of t i.e the ow is steady, $\dfrac{\partial P}{\partial x}=0$ because the air pressure is constant and the x component of the body force is $b_x=g\sin\theta$. Navier Stokes equation becomes;

$$0=0+g\sin\theta+\nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial y^2}=-\frac{g\sin\theta}{\nu}$$

Integrating twice on y: $u=-\dfrac{g(\sin\theta)y^2}{2\nu}+c_1 y+c_2$ (3.0.5)

and applying the boundary conditions that $u=0$, at $y=0$, we obtain;

$$c_2 = 0$$

differentiating the equation (3.0.5) partially with respect to y, we obtain,

$$\frac{\partial u}{\partial y} = -\frac{g(\sin\theta)y}{\nu} + c_1$$

and applying the boundary conditions that $\tau_{xy} = \mu \frac{\partial u}{\partial y} = 0$ at $y = \delta$,we obtain;

$$c_1 = \frac{g(\sin\theta)\delta}{\nu}$$

This enables us to find the velocity distribution $u(y)$

By substituting for c_1 and c_2 into equation (3.0.5), we obtain:

$$u = \frac{g\sin\theta}{\nu}\left(\delta y - \frac{y^2}{2}\right) \qquad (3.0.6)$$

3.1 Momentum integral equation

Now, employ the method of momentum integral equation in determining the boundary layer thickness, δ. The boundary layer equations in two-dimensions for steady incompressible flow are given by Mohanty (1994):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

Consider the Figure 3.1 below:

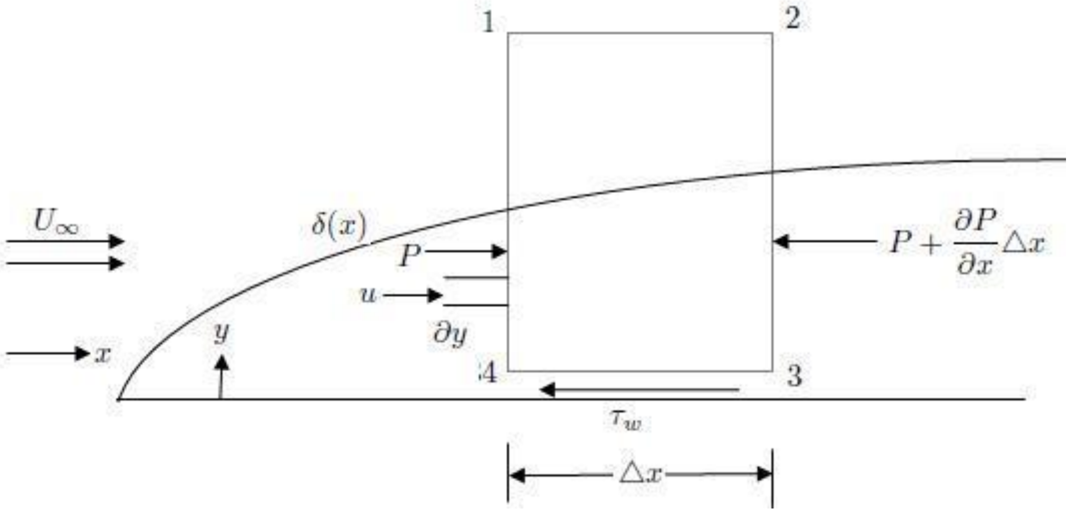

Figure 3.1: Control Volume Analysis of the Boundary Layer

A free stream flow at U_∞ approaches a surface whose leading edge coincides with $x = 0$. x is measured along the surface and y perpendicular to it. $\delta(x)$ is the thickness of the boundary layer at location x. 1-2-3-4 define a control volume whose faces 1-2 and 3-4 are parallel to the solid surface and the other two faces perpendicular to the surface. The height of the face 1-4 or 2-3 is ℓ and ℓ is greater than the thickness of the boundary layer.

Fluid masses enter through faces 1-4, 1-2 and 2-3 carrying with them the momentum prevailing in the respective neighbourhood. No mass enters through 3-4, the face being coincident with the solid wall. The face 3-4, on the other hand, experiences the wall shear stress and is Δx long. A unit depth perpendicular to the plane of 1-2-3-4 is being considered.

3.2 Conservation of Momentum

The momentum in-flow through a strip dy is $\rho u^2 dy$, and through the face 1-4 is;

$$\int_0^\delta \rho u^2 dy \qquad (3.2.1)$$

Outflow through 2-3 is

$$\int_0^{\ell} \rho u^2 dy + \frac{d}{dx}\left(\int_0^{\ell} \rho u^2 dy\right)\Delta x \qquad (3.2.2)$$

inflow through 1-2 due to the mass coming from the zone U_∞ is:

$$U_\infty \frac{d}{dx}\left(\int_0^{\ell} \rho u dy\right)\Delta x \qquad (3.2.3)$$

Combining (3.2.1),(3.2.2) and (3.2.3), the net flux of momentum through the control surface becomes:

$$\frac{d}{dx}\left(\int_0^{\ell} \rho u^2 dy\right)\Delta x - U_\infty \frac{d}{dx}\left(\int_0^{\ell} \rho u dy\right)\Delta x \qquad (3.2.4)$$

The face 1-2 being in the free stream zone, no shear stress acts on it. The pressure on the face 1-4 is P, and is independent of y by boundary layer theory Schlichting (1955).

The external forces acting on the control volume are hence;

$$-\tau_w \Delta x - \frac{dP}{dx}\Delta x.\ell \qquad (3.2.5)$$

Combining expressions (3.2.4) and (3.2.5), write the momentum balance for the control volume as;

$$U_\infty \frac{d}{dx}\left(\int_0^{\ell} \rho u dy\right)\Delta x - \frac{d}{dx}\left(\int_0^{\ell} \rho u^2 dy\right)\Delta x = \left(\tau_w \Delta x + \frac{dP}{dx}.\int_0^{\ell} dy\right)\Delta x \qquad (3.2.6)$$

In order to evaluate the pressure gradient we can move into the free stream zone and use Bernoulli's equation McClamroch (2011),

$$\frac{P}{\rho} + \frac{U_\infty^2}{2} = C \qquad (3.2.7)$$

or

$$\frac{dP}{dx} = -\rho U_\infty \frac{dU_\infty}{dx} \qquad (3.2.8)$$

Equation (3.2.6) is now written using equation (3.2.8)

$$U_\infty \frac{d}{dx}\left(\int_0^\ell \rho u dy\right) + \rho U_\infty \frac{dU_\infty}{dx}\int_0^\ell dy - \frac{d}{dx}\left(\int_0^\ell \rho u^2 dy\right) = \tau_w \qquad (3.2.9)$$

or for incompressible flow

$$U_\infty \frac{d}{dx}\left(\int_0^\ell u dy\right) + U_\infty \frac{dU_\infty}{dx}\int_0^\ell dy - \frac{d}{dx}\left(\int_0^\ell u^2 dy\right) = \frac{\tau_w}{\rho} \qquad (3.2.10)$$

Consider the differentiation

$$\frac{d}{dx}\left(U_\infty \int_0^\ell u dy\right) = U_\infty \frac{d}{dx}\int_0^\ell u dy + \frac{dU_\infty}{dx}\int_0^\ell u dy$$

or

$$U_\infty \frac{d}{dx}\int_0^\ell u dy = \frac{d}{dx}\int_0^\ell u U_\infty dy - \frac{dU_\infty}{dx}\int_0^\ell u dy \qquad (3.2.11)$$

Thus Equation (3.2.10) can be rewritten as

$$\left(\frac{d}{dx}\int_0^\ell u U_\infty dy - \frac{d}{dx}\int_0^\ell u^2 dy\right) + \frac{dU_\infty}{dx}\left(\int_0^\ell U_\infty dy - \int_0^\ell u dy\right) = \frac{\tau_w}{\rho}$$

or

$$\frac{d}{dx}\int_0^\ell \left(u U_\infty - u^2\right) dy + \frac{dU_\infty}{dx}\int_0^\ell \left(U_\infty - u\right) dy = \frac{\tau_w}{\rho} \qquad (3.2.12)$$

The limits of integration 0 to ℓ can be split up into 0 to δ and δ to ℓ. In the free stream region of δ to ℓ, however, $u = U_\infty$ and each of the integrand is zero. Hence equation (3.2.12) is effectively

$$\frac{d}{dx}\int_0^\delta \left(u U_\infty - u^2\right) dy + \frac{dU_\infty}{dx}\int_0^\delta \left(U_\infty - u\right) dy = \frac{\tau_w}{\rho} \qquad (3.2.13)$$

In the present form of equation (3.2.13), the integral equation for momentum can represent both laminar and turbulent flows, since no assumption has yet been made for the shear stress, τ_w.

3.3 Solution of the momentum integral equation

The steps involved in solving equation (3.2.13) are:

1. choosing a velocity profile that satisfies all the essential and some additional boundary conditions,
2. evaluating the integrals and reducing the left hand side to a differential expression on δ,
3. postulating the law of shear stress for τ_w, depending on the flow regime, for laminar flow $\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$ by Newton's law of shear stress and
4. solving the differential equation for δ.

Consider an incompressible, laminar, steady flow of a Newtonian fluid along an inclined plane at zero incidences.

3.4 Estimates of Boundary Layer Velocity Profiles

The essential conditions to be satisfied by the boundary layer velocity profile are:

1. $y = 0, \quad u = 0, \quad v = 0$ no slip on the wall.
2. $y = \delta, \quad u = U_\infty$ free stream velocity at the edge of boundary layer.
3. $y = \delta, \quad \mu\frac{\partial u}{\partial y} = 0$ no shear stress at the edge of the boundary layer.

It is then proposed that the boundary layer velocity profile can be written in terms of a polynomial in y with the number of terms that are equal the number of boundary conditions to be satisfied. For the first two conditions listed above, choose linear polynomial velocity profile approximation:

$$u_1 = A_1 + B_1 y \tag{3.4.1}$$

For the three conditions listed above, choose quadratic polynomial velocity profile approximation:

$$u_2 = A_2 + B_2 y + C_2 y^2 \tag{3.4.2}$$

and sine function velocity profile approximation:

$$u_3 = A_3 \sin B_3 y \tag{3.4.3}$$

The velocity profiles approximations u_1, u_2, u_3 which are required to find the boundary layer thickness δ of flow with net flow is to be obtained, then use it to find the effect of velocity by varying angles of inclination between $0 < \theta < \dfrac{\pi}{2}$.

3.4.1 Velocity profile approximated as a linear Polynomial

It can be proposed that the boundary layer velocity profile can be written in terms of a linear polynomial in y as given in equation (3.4.1). where A_1 and B_1 are real numbers. Using the essential conditions 1 and 2 in section 3.4, then:

1. $A_1 = 0$ using condition 1.
2. $U_\infty = B_1 \delta$ using condition 2.

Solution of the two algebraic equations yield; $A_1 = 0$, $B_1 = \dfrac{U_\infty}{\delta} 1$ Substituting for A_1 and B_1 in equation (3.4.1), then the velocity profile is:

$$u_1 = \frac{U_\infty}{\delta} y \quad \text{or} \quad \frac{u_1}{U_\infty} = \frac{y}{\delta} \tag{3.4.4}$$

3.4.2 Velocity Profile approximated as a quadratic Polynomial

It can also be proposed that the boundary layer velocity profile can be written in terms of a quadratic polynomial in y as given in equation (3.4.2). Differentiating equation (3.4.2), partially with respect to y,

$$\frac{\partial u_2}{\partial y} = B_2 + 2C_2 y \tag{3.4.5}$$

where A_2, B_2, and C_2 are real numbers. Using the conditions 1, 2 and 3 given in section 3.4, in equations (3.4.2) and (3.4.5), then:

1. $A_2 = 0$ using condition 1.
2. $U_\infty = B_2\delta + C_2\delta^2$ using condition 2.
3. $0 = B_2 + C_2\delta$ using condition 3

Solution of the two algebraic equations yield; $A_2 = 0$, $B_2 = 2\frac{U_\infty}{\delta}$, $C_2 = -\frac{U_\infty}{\delta^2}$

Substituting for A_2, B_2, and C_2 in equation (3.4.2), then the velocity profile is:

$$u_2 = 2\frac{U_\infty}{\delta}y - \frac{U_\infty}{\delta^2}y^2 \text{ or } \frac{u_2}{U_\infty} = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$$

3.4.3 Velocity Profile approximated as a sine function

It is then proposed that the boundary layer velocity profile can be written as a sine function as given in equation (3.4.3). $\frac{\partial u_3}{\partial y} = A_3 B_3 \cos B_3 y$ and $\frac{\partial^2 u_3}{\partial y^2} = -A_3 B_3^2 \sin By$

where A_3 and B_3 are real numbers.

Using the essential conditions 1, 2, and 3 in section 3.4 , then:

1. $0 = A_3 \sin 0$ using condition 1,
2. $U_\infty = A_3 \sin B_3\delta$ using condition 2, and

3. $0 = A_3 B_3 \cos B_3 \delta$ using condition 3.

It is clear that using condition 1 do not give us significant results. Solution of the two algebraic equations i.e 2 and 3 yield; $A_3 = U_\infty$, and $B_3 = \frac{\pi}{2\delta}$. Substituting for A_3 and B_3 in equation (3.4.3), then the velocity profile is;

$$u_3 = U_\infty \sin \frac{\pi}{2\delta} y \text{ or } \frac{u_3}{U_\infty} = \sin \frac{\pi}{2\delta} y \qquad (3.4.7)$$

3.5 Boundary layer thickness

Find the expressions to approximate the values of boundary layer. Since U_∞ is not varying, it is independent of x, then $\frac{dU_\infty}{dx} = 0$.

Equation (3.2.13) can be reduced to

$$\frac{d}{dx}\int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy = \frac{\tau_w}{U_\infty^2} \qquad (3.5.1)$$

3.5.1 Boundary layer thickness from the velocity profile approximated as a linear Polynomial

The integral equation (3.5.1) can also be evaluated by substituting for $\frac{u_1}{U_\infty}$ from equation (3.4.4), and then;

$$\frac{d}{dx}\int_0^\delta \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right) dy = \frac{\tau_w}{U_\infty^2 \rho}$$

$$\frac{d}{dx}\left(\int_0^\delta \frac{y}{\delta} dy - \int_0^\delta \frac{y^2}{\delta^2} dy\right) = \frac{\tau_w}{U_\infty^2 \rho}$$

$$\frac{1}{6}\frac{d\delta}{dx} = \frac{\tau_w}{U_\infty^2 \rho} \qquad (3.5.2)$$

Now impose the laminar flow condition, for which

$$\tau_w = \mu\left(\frac{\partial u_1}{\partial y}\right)_{y=0} \tag{3.5.3}$$

Now,

$$u_1 = \frac{U_\infty}{\delta} y,$$

$$\frac{\partial u_1}{\partial y} = \frac{U_\infty}{\delta} \tag{3.5.4}$$

$$\left(\frac{\partial u_1}{\partial y}\right)_{y=0} = \frac{U_\infty}{\delta} \tag{3.5.5}$$

hence,

$$\tau_w = \mu\frac{U_\infty}{\delta} \tag{3.5.6}$$

And

$$\frac{\tau_w}{U_\infty^2 \rho} = \frac{\mu}{\rho U_\infty \delta} \tag{3.5.7}$$

Substituting equation (3.5.7) into equation (3.5.2), then

$$\frac{1}{6}\frac{d\delta}{dx} = \frac{\mu}{\rho U_\infty \delta} \tag{3.5.8}$$

or

$$\frac{1}{6}\delta d\delta = \frac{\nu}{U_\infty} dx \tag{3.5.9}$$

integrating equation (3.5.9), we obtain,

$$\delta = \left(12\frac{\nu}{U_\infty} x\right)^{\frac{1}{2}} + C \tag{3.5.10}$$

Let this boundary layer thickness δ be denoted by δ_1.

On the surface at $x = 0$ and $\delta_1 = 0$, $C = 0$,

$$\delta_1 = \left(12\frac{\nu}{U_\infty}x\right)^{\frac{1}{2}} \quad \text{or} \quad \delta_1 = \left(\frac{12}{\mathrm{Re}}\right)^{\frac{1}{2}} x \qquad (3.5.11)$$

Where $\mathrm{Re} = \frac{U_\infty}{\nu}$.

3.5.2 Boundary layer thickness from the velocity profile approximated as a quadratic polynomial

The integral equation (3.5.1) can also be evaluated by substituting for $\frac{u_2}{U_\infty}$ from equation (3.4.6), and then;

$$\frac{d}{dx}\int_0^\delta \left(2\frac{y}{\delta}-\frac{y^2}{\delta^2}\right)\left[1-\left(2\frac{y}{\delta}-\frac{y^2}{\delta^2}\right)\right]dy=\frac{\tau_w}{U_\infty^2\rho}$$

$$\frac{d}{dx}\int_0^\delta \left(2\frac{y}{\delta}-5\frac{y^2}{\delta^2}+4\frac{y^3}{\delta^3}-\frac{y^4}{\delta^4}\right)dy=\frac{\tau_w}{U_\infty^2\rho}$$

$$\frac{2}{15}\frac{d\delta}{dx}=\frac{\tau_w}{U_\infty^2\rho} \qquad (3.5.13)$$

Now impose the laminar flow condition, for which

$$\tau_w=\mu\left(\frac{\partial u_2}{\partial y}\right)_{y=0} \qquad (3.5.13)$$

Now,

$$u_2=2\frac{U_\infty}{\delta}y-\frac{U_\infty}{\delta^2}y^2,$$

$$\frac{du_2}{dy}=2\frac{U_\infty}{\delta}-2\frac{U_\infty}{\delta^2}y \qquad (3.5.14)$$

$$\left(\frac{\partial u_2}{\partial y}\right)_{y=0}=2\frac{U_\infty}{\delta} \qquad (3.5.15)$$

hence,

$$\tau_w=2\mu\frac{U_\infty}{\delta} \qquad (3.5.16)$$

and

$$\frac{\tau_w}{U_\infty^2\rho}=2\frac{\mu}{\rho U_\infty\delta} \qquad (3.5.17)$$

Substituting equation (3.5.17) into equation (3.5.12), then

$$\frac{2}{15}\frac{d\delta}{dx} = 2\frac{\mu}{\rho U_\infty \delta} \tag{3.5.18}$$

or

$$\frac{2}{15}\delta d\delta = 2\frac{\mu}{U_\infty \rho}dx \tag{3.5.19}$$

integrating the differential equation (3.5.19), we obtain,

$$\frac{1}{15}\delta^2 = 2\frac{\mu}{U_\infty \rho}x + C \tag{3.5.20}$$

Let this boundary layer thickness δ be denoted by δ_2.

On the surface at $x = 0$ and $\delta_2 = 0$, $C = 0$, then,

$$\delta_2 = \left(30\frac{\nu}{U_\infty}x\right)^{\frac{1}{2}} \quad \text{or} \quad \delta_2 = \left(\frac{30}{\mathrm{Re}}\right)^{\frac{1}{2}} x \tag{3.5.21}$$

Where $\mathrm{Re} = \dfrac{U_\infty}{\nu}$.

3.5.3 Boundary layer thickness from the velocity profile approximated as a sine function

The integral equation (3.5.1) can also be evaluated by substituting for $\dfrac{u_3}{U_\infty}$ from equation (3.4.7), and then;

$$\frac{d}{dx}\int_0^\delta \sin\frac{\pi}{2\delta}y\left(1-\sin^2\frac{\pi}{2\delta}y\right)dy = \frac{\tau_w}{U_\infty^2\rho}$$

$$\frac{d}{dx}\int_0^\delta \left(\sin\frac{\pi}{2\delta}y - \sin^2\frac{\pi}{2\delta}y\right)dy = \frac{\tau_w}{U_\infty^2\rho}$$

$$\frac{d}{dx}\left[\int_0^\delta \sin\frac{\pi}{2\delta}ydy - \frac{1}{2}\int_0^\delta\left(1-\cos\frac{\pi}{\delta}y\right)dy\right] = \frac{\tau_w}{U_\infty^2\rho}$$

$$\left(\frac{4-\pi}{2\pi}\right)\frac{d\delta}{dx}=\frac{\tau_w}{U_\infty^2\rho} \qquad (3.5.23)$$

Now impose the laminar flow condition, for which

$$\tau_w=\mu\left(\frac{\partial u_3}{\partial y}\right)_{y=0} \qquad (3.5.24)$$

Now,

$$u_3=U_\infty \sin\frac{\pi y}{2\delta},$$

$$\frac{du_3}{dy}=\frac{U_\infty \pi}{2\delta}\cos\frac{\pi y}{2\delta} \qquad (3.5.25)$$

$$\left(\frac{\partial u_3}{\partial y}\right)_{y=0}=U_\infty\frac{\pi}{2\delta} \qquad (3.5.26)$$

hence,

$$\tau_w=\frac{\mu U_\infty \pi}{2\delta} \qquad (3.5.27)$$

and

$$\frac{\tau_w}{U_\infty^2\rho}=\frac{\pi\nu}{2U_\infty\delta} \qquad (3.5.28)$$

Substituting equation (3.5.28) into equation (3.5.23), then

$$\left(\frac{4-\pi}{2\pi}\right)\frac{d\delta}{dx}=\frac{\pi\nu}{2U_\infty\delta} \qquad (3.5.29)$$

or

$$\left(\frac{4-\pi}{2\pi}\right)\delta d\delta=\frac{\pi\nu}{2U_\infty}dx \qquad (3.5.30)$$

integrating the differential equation (3.5.30), we obtain,

$$\delta = \left(\frac{2\pi^2 \nu x}{(4-\pi)U_\infty} \right)^{\frac{1}{2}} + C \qquad (3.5.31)$$

Let this boundary layer thickness δ be denoted by δ_3.

On the surface at $x = 0$ and $\delta_3 = 0$, $C = 0$, then,

$$\delta_3 = \left(\frac{2\pi^2 \nu x}{(4-\pi)U_\infty} \right)^{\frac{1}{2}} \qquad (3.5.32)$$

$$\delta_3 = \left(\frac{2}{(4-\pi)\,\mathrm{Re}} \right)^{\frac{1}{2}} \pi x \qquad (3.5.33)$$

Where $\mathrm{Re} = \frac{U_\infty}{\nu}$.

3.6 Velocity down Inclined Plane

The expressions for the velocity down inclined plane at various angles are obtained by substituting the expressions of δ obtained in section 3.5 above into equation (3.0.6). The velocity u_1 down inclined plane using linear velocity profile can be approximated by substituting equation (3.5.11) into equation (3.0.6) to give,

$$u_1 = \frac{b \sin\theta}{\nu} \left[\left(\frac{12}{\mathrm{Re}} \right)^{\frac{1}{2}} xy - \frac{y^2}{2} \right] \qquad (3.6.1)$$

The velocity u_2 down inclined plane using quadratic velocity profile can be approximated by substituting equation (3.5.22) into equation (3.0.6) to give,

$$u_2 = \frac{b \sin\theta}{\nu} \left[\left(\frac{30}{\mathrm{Re}} \right)^{\frac{1}{2}} xy - \frac{y^2}{2} \right] \qquad (3.6.2)$$

The velocity u_3 down inclined plane using sine function velocity profile can be approximated by substituting equation (3.5.33) into equation (3.0.6) to give,

$$u_3 = \frac{b\sin\theta}{\nu}\left[\left(\frac{2}{(4-\pi)\mathrm{Re}}\right)^{\frac{1}{2}}\pi xy - \frac{y^2}{2}\right] \qquad (3.6.3)$$

CHAPTER FOUR

4.0 RESULTS AND DISCUSSION

4.1 Results

Taking water as an illustration at temperature $5^0 C$, then $\nu = 1.519 \times 10^{-6} m^2/s$, Labye (1988); $g = 9.8 m/s^2$, $\rho = 1000\, kg/m^3$, $U_\infty = 1 m/s$.

Now the Reynolds number(Re) for laminar ow is between 0 and 2300, and $\mathrm{Re} = \dfrac{U_\infty x}{\nu}$. Find $x = \dfrac{\nu \mathrm{Re}}{U_\infty}$ such that Re falls between 0 and 2300. x is the distance from the point where the fluid from the main stream meets the inclined plane, and along the surface of the inclined plane. Compare results for u when δ used was obtained by momentum integral approach at various angles of inclination between $0 < \theta < \dfrac{\pi}{2}$ for linear, quadratic and sine function velocity profiles.

δ_1, δ_2, and δ_3 are the boundary layer thickness approximations from the , linear, quadratic and sine function velocity profiles respectively are substituted into momentum integral equation. u_1, u_2, and u_3 are the velocities down inclined plane when δ_1, δ_2, and δ_3 respectively are substituted into equation (3.0.6).

The results using the data below are given in the respective tables and represented in respective Figures.

1. $x = 0.0005\, m$, $\mathrm{Re} = 0.3292 \times 10^3$,

   $\delta_1 = 4.4721 \times 10^{-5}$,

$\delta_2 = 7.0711 \times 10^{-5}$,

$\delta_3 = 6.1907 \times 10^{-5}$,

Table 1: Velocity and the angles of inclination (x=0.0005m)

θ	$u_1(m/s)$	$u_2(m/s)$	$u_3(m/s)$
0	0	0	0
$\frac{\pi}{12}$	0.0017	0.0042	0.0032
$\frac{\pi}{6}$	0.0032	0.0081	0.0062
$\frac{\pi}{4}$	0.0046	0.0114	0.0087
$\frac{\pi}{3}$	0.0056	0.0140	0.0107
$\frac{5\pi}{12}$	0.0062	0.0156	0.0119

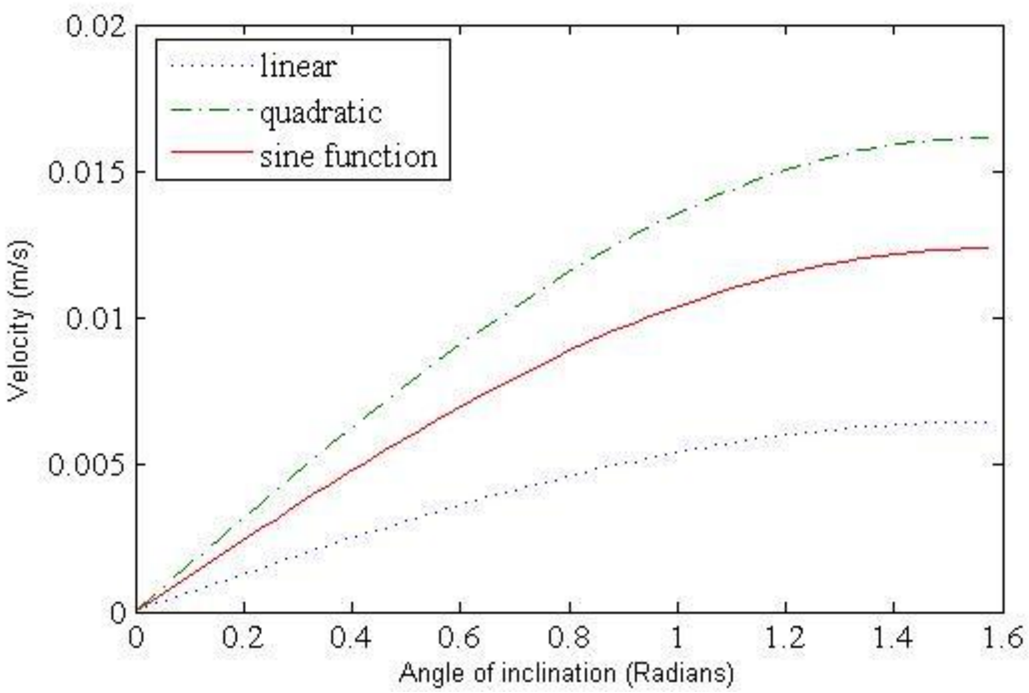

Figure 4.1: Velocity versus angle of inclination when $x = 0.0005\text{m}$

2. $x = 0.0010\, m, \quad \mathrm{Re} = 0.6583 \times 10^{3},$

$\delta_1 = 8.9443 \times 10^{-5},$

$\delta_2 = 1.4142 \times 10^{-4},$

$\delta_3 = 1.2381 \times 10^{-4},$

Table 2: Velocity and the angles of inclination (x=0.0010m)

θ	$u_1(m/s)$	$u_2(m/s)$	$u_3(m/s)$
0	0	0	0
$\frac{\pi}{12}$	0.0067	0.0167	0.0128
$\frac{\pi}{6}$	0.0129	0.0323	0.0247
$\frac{\pi}{4}$	0.0182	0.0456	0.0350
$\frac{\pi}{3}$	0.0223	0.0559	0.0428
$\frac{5\pi}{12}$	0.0249	0.0623	0.0478

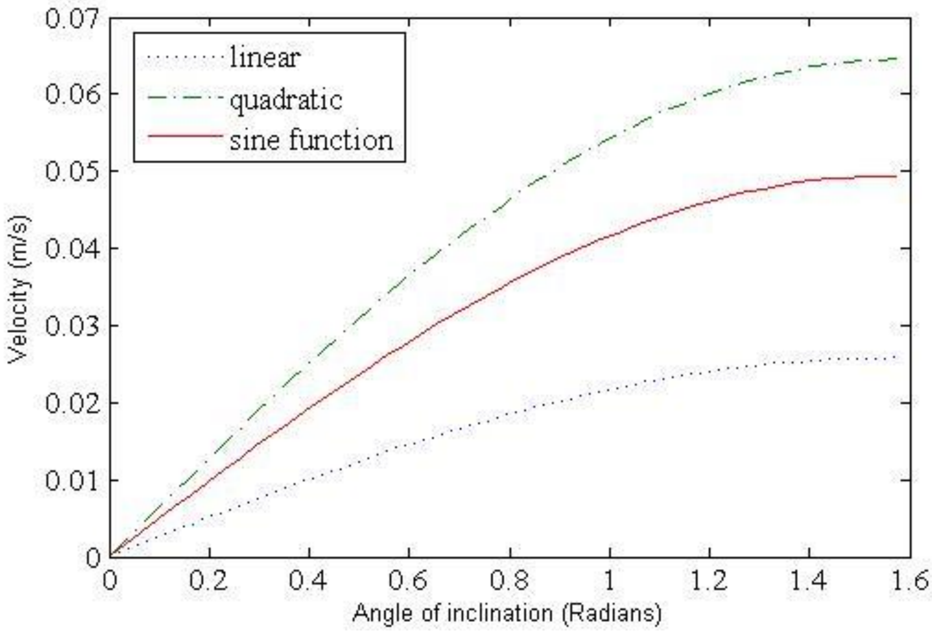

Figure 4.2: Velocity versus angle of inclination when $x = 0.0010\mathrm{m}$

3. $x = 0.0015\, m, \quad \mathrm{Re} = 0.9875 \times 10^{3}$,

$\delta_1 = 1.3416 \times 10^{-4}$,

$\delta_2 = 2.1213 \times 10^{-4}$,

$\delta_3 = 1.8572 \times 10^{-4}$,

Table 3: Velocity and the angles of inclination (x=0.0015m)

θ	$u_1(m/s)$	$u_2(m/s)$	$u_3(m/s)$
0	0	0	0
$\frac{\pi}{12}$	0.0150	0.0376	0.0288
$\frac{\pi}{6}$	0.0290	0.0726	0.0556
$\frac{\pi}{4}$	0.0411	0.1026	0.0787
$\frac{\pi}{3}$	0.0503	0.1257	0.0964
$\frac{5\pi}{12}$	0.0561	0.1402	0.1075

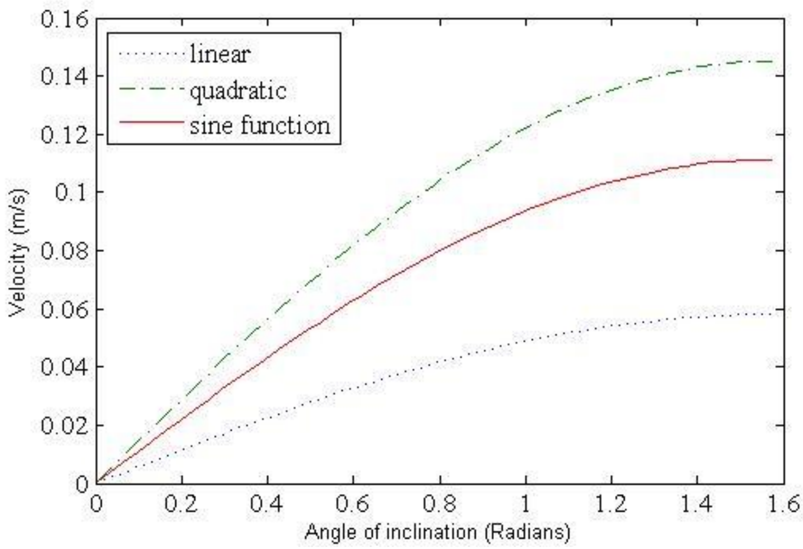

Figure 4.3: Velocity versus angle of inclination when $x = 0.0015\mathrm{m}$

4. $x = 0.0020\, m, \quad \text{Re} = 1.3167 \times 10^{3},$

$\delta_1 = 1.7889 \times 10^{-4},$

$\delta_2 = 2.8284 \times 10^{-4},$

$\delta_3 = 2.4763 \times 10^{-4},$

Table 4: Velocity and the angles of inclination (x=0.0020m)

θ	$u_1(m/s)$	$u_2(m/s)$	$u_3(m/s)$
0	0	0	0
$\frac{\pi}{12}$	0.0267	0.0668	0.0512
$\frac{\pi}{6}$	0.0516	0.1290	0.0989
$\frac{\pi}{4}$	0.0730	0.1825	0.1399
$\frac{\pi}{3}$	0.0894	0.2235	0.1713
$\frac{5\pi}{12}$	0.0997	0.2493	0.1911

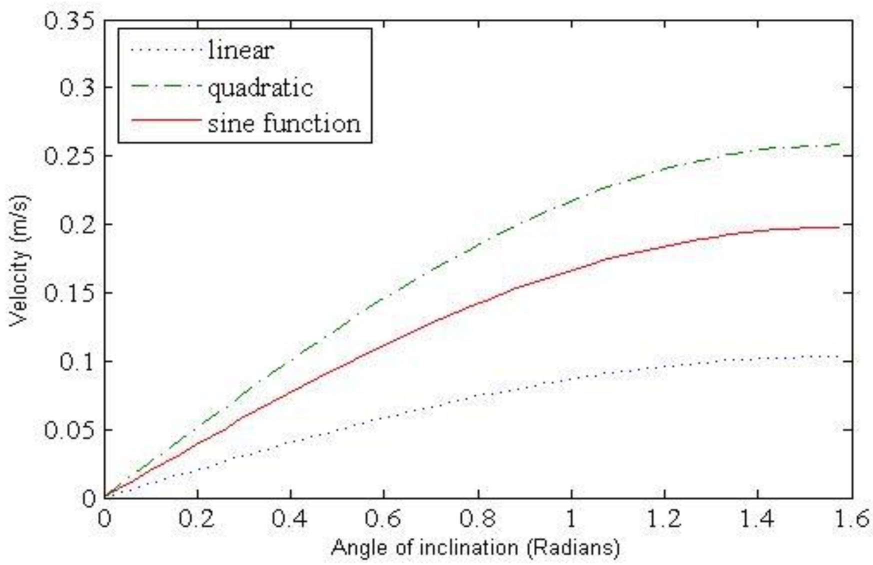

Figure 4.4: Velocity versus angle of inclination when $x = 0.0020m$

5. $x = 0.0025\, m, \quad \text{Re} = 1.6458 \times 10^{3},$

$\delta_1 = 2.2361 \times 10^{-4}$,

$\delta_2 = 3.5355 \times 10^{-4}$,

$\delta_3 = 3.0954 \times 10^{-4}$,

Table 5: Velocity and the angles of inclination (x=0.0025m)

θ	$u_1(m/s)$	$u_2(m/s)$	$u_3(m/s)$
0	0	0	0
$\frac{\pi}{12}$	0.0417	0.1044	0.0800
$\frac{\pi}{6}$	0.0806	0.2016	0.1545
$\frac{\pi}{4}$	0.1140	0.2851	0.2185
$\frac{\pi}{3}$	0.1397	0.3492	0.2677
$\frac{5\pi}{12}$	0.1558	0.3895	0.2985

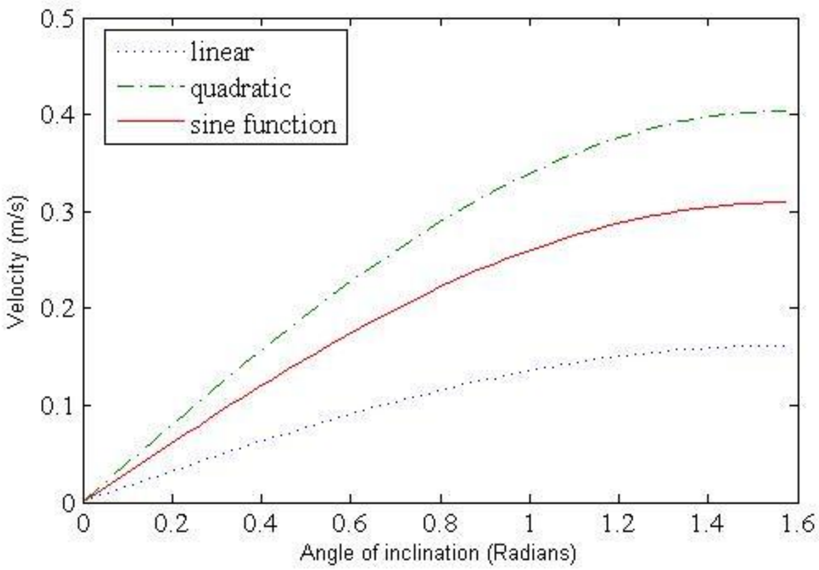

Figure 4.5: Velocity versus angle of inclination when $x = 0.0025m$

6. $x = 0.0030\, m$, $\mathrm{Re} = 1.9750 \times 10^{3}$,

$\delta_1 = 2.6833 \times 10^{-4}$,

$\delta_2 = 4.2426 \times 10^{-4}$,

$\delta_3 = 3.7144 \times 10^{-4}$,

Table 6: Velocity and the angles of inclination (x=0.0030m)

θ	$u_1(m/s)$	$u_2(m/s)$	$u_3(m/s)$
0	0	0	0
$\frac{\pi}{12}$	0.0601	0.1503	0.1152
$\frac{\pi}{6}$	0.1161	0.2903	0.2225
$\frac{\pi}{4}$	0.1642	0.4106	0.3147
$\frac{\pi}{3}$	0.2011	0.5029	0.3854
$\frac{5\pi}{12}$	0.2243	0.5609	0.4299

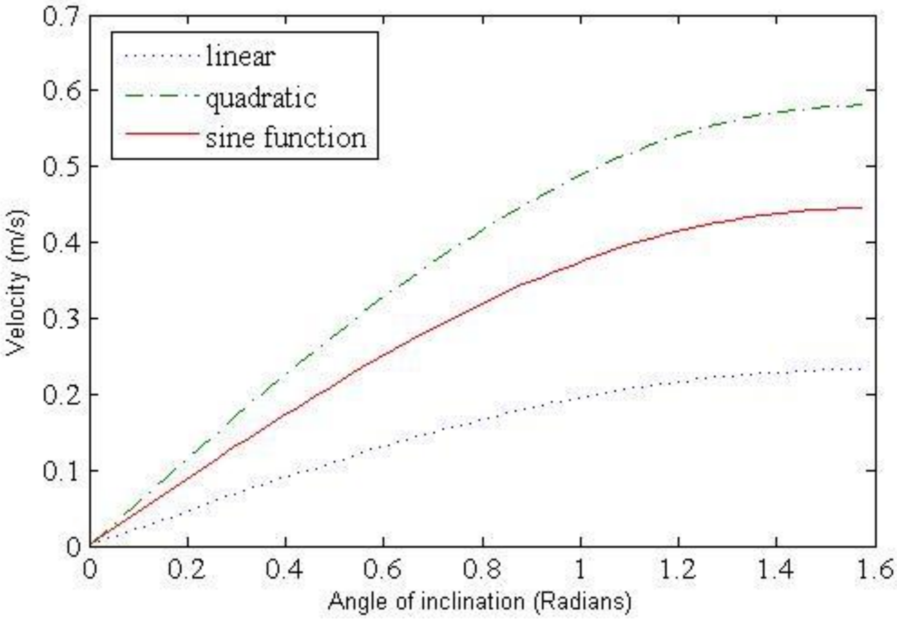

Figure 4.6: Velocity versus angle of inclination when $x = 0.0030m$

4.2 Discussion

Momentum integral approach has been used to obtain the boundary layer thickness, δ. Linear and quadratic polynomial approximations and sine function approximation of the velocity profile in boundary layer were obtained. This approximation velocity profiles are used in the momentum integral equation to generate the boundary layer thickness. Considering the illustration above (section 4.1), it demonstrates the results that the velocities as a result of the linear, and quadratic polynomials and sine function approximation velocity profiles were used. The results are generated with the help of MatLab. From Table 1 to Table 6 and their corresponding graphs respectively, δ increases from zero the point where Re is 2300, the boundary layer thickness δ_1, δ_2, and δ_3, and u_1, u_2, and u_3 also increases respectively for the three velocity profiles. This is because of molecular interactions that generate viscous forces. At the surface, the flow has zero relative speed, because the fluid seems to stick to the surface of the inclined plane due to adhesive forces. The fluid transfers momentum to the adjacent layers through the action of viscosity. This leads to increase in velocity and the boundary layer thickness as x increases.

Comparing the velocities down inclined plane for the three velocity profiles, the velocity u_1 is lower than u_2 and u_3, followed by u_3 and u_2 is the highest. This is after taking into consideration various angles of inclination between $0 < \theta < \frac{\pi}{2}$. To obtain δ_1 and δ_3 which result in u_1 and u_3 respectively, only two boundary layer conditions are considered. To obtain δ_2 which results in u_2, the three boundary layer conditions are utilized. This gives the reason that this approximation gives higher velocities than the other approximations. For laminar flow, it is expected that the velocities do not go beyond $1m/s$ i.e main stream velocity but approach it.

CHAPTER FIVE

5.0 CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

Momentum integral approach gives an approximation of boundary layer thickness, δ. The linear and quadratic polynomials, and sine function approximation of velocity profiles, leads us to obtain the values of boundary layer thickness using the momentum integral equation. These values are used at every angle of inclination between $0 < \theta < \frac{\pi}{2}$ to obtain the velocity down an inclined plane. It is preferable to use the quadratic polynomial velocity profile than the linear polynomial, and sine function approximation of velocity profiles. This is because quadratic polynomial velocity profile utilizes more boundary conditions, hence more reliable.

5.2 Recommendations

1. Momentum integral approach has been used to obtain δ. Numerical methods like finite elements method and Finite difference method can be used to obtain δ
2. The velocity profiles approximations of polynomials of degrees 1 and 2, and sine function have been considered. Other polynomials of higher degrees can be considered.
3. The coefficient friction of the inclined plane can also be put into consideration.

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