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On almost similarity results on partial isometrics, θ-operators and posinormal operators

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Abstract
It is pre-eminent that equality of spectra is realized when the two given operators are unitarily equivalent or similar but not when they are almost similar. Also, projections which are α-almost similar, demonstrate that under certain conditions, they not only have equal spectra but also equal approximate point spectra. Though almost similarity property has been studied mostly, there is still gap in literature linking it directly to commuting condition for partial isometries, θ-operators and posinormal operators. This paper exhibits some primary results of such nature on the above mentioned classes of operators.

Keywords: Almost similarity, θ-operators and posinormal operators

Introduction
In this paper, complete normed linear space is denoted H while $\mathfrak{B}(H)$ denotes the Banach algebra of bounded linear operators on H. $\mathcal{A}$ and $\mathcal{B}$ denotes operators on $\mathfrak{B}(H)$. According to Jibril (1996), two operators $\mathcal{A}, \mathcal{B} \in \mathfrak{B}(H)$ are said to be almost similar, denoted as $\mathcal{A} \overset{a.s}{\sim} \mathcal{B}$ if it exists an invertible operator $\mathcal{N}$ such that $\mathcal{A}^*\mathcal{A} = \mathcal{N}^{-1}\mathcal{B}^*\mathcal{B}\mathcal{N}$ and $\mathcal{A}^* + \mathcal{A} = \mathcal{N}^{-1}(\mathcal{B}^* + \mathcal{B})\mathcal{N}$ hold. This property has been extensively researched on and variety of results demonstrated.

An operator $\mathcal{A} \in \mathfrak{B}(H)$ is said to be:

- Self-adjoint (hermitian) if $\mathcal{A}^* = \mathcal{A}$
- Isometry if $\mathcal{A}^*\mathcal{A} = I$
- Partially Isometric if $\mathcal{A}\mathcal{A}^*\mathcal{A} = \mathcal{A}$
- Coisometry if $\mathcal{A}\mathcal{A}^* = I$
- Unitary if $\mathcal{A}^*\mathcal{A} = \mathcal{A}\mathcal{A}^* = I$
- Orthogonal projection if $\mathcal{A}^2 = I$
- Posinormal if $\mathcal{A}\mathcal{A}^* = \mathcal{A}^*P\mathcal{A}$, for $P \geq 0$
- θ-operator ($\mathcal{A} \in \theta$) if $[\mathcal{A}^*\mathcal{A}, \mathcal{A}^* + \mathcal{A}] = 0$ i.e $\mathcal{A}^*\mathcal{A}$ and $\mathcal{A}^* + \mathcal{A}$ commute.

Main Results
Theorem
Let $\mathcal{A}, \mathcal{B} \in \mathfrak{B}(H)$ such that $\mathcal{A} \overset{a.s}{\sim} \mathcal{B}$. If $\mathcal{A}^2$ is a partial isometry and $\mathcal{B}$ is self-adjoint, then $\mathcal{B}^2$ is also partially isometric.

Proof
Since $\mathcal{A}^2$ is a partial isometry, we have $\mathcal{A}^2 = \mathcal{A}^2\mathcal{A}^{*2}\mathcal{A}^2$ and by projection property, we also have that $\mathcal{A}\mathcal{A}^* = \mathcal{A}\mathcal{A} = \mathcal{A}^2$.

$\mathcal{A} \overset{a.s}{\sim} \mathcal{B}$, implies there exists an invertible operator $\mathcal{N}$ such that

$$\mathcal{B}^*\mathcal{B} = \mathcal{N}^{-1}\mathcal{A}^*\mathcal{A}\mathcal{N} \quad (1)$$

and

$$\mathcal{B}^* + \mathcal{B} = \mathcal{N}^{-1}(\mathcal{A}^* + \mathcal{A})\mathcal{N} \quad (2)$$

From $\mathcal{A}\mathcal{A}^* = \mathcal{A}\mathcal{A} = \mathcal{A}^2$, then (1) becomes,

$\mathcal{B}^*\mathcal{B} = \mathcal{N}^{-1}\mathcal{A}^2\mathcal{N}$.

Consequently, $\mathcal{B}^*\mathcal{B} = \mathcal{B}^2$ and thus, $\mathcal{B}^2 = \mathcal{N}^{-1}\mathcal{A}^2\mathcal{N}$. It follows that, $\mathcal{B}^2 = \mathcal{B}^2\mathcal{B}^{*2}\mathcal{B}^2$, which is equivalent to $\mathcal{B}^2 - \mathcal{B}^2\mathcal{B}^{*2}\mathcal{B}^2 = 0$ $\mathcal{B}^2(1 - \mathcal{B}^{*2}\mathcal{B}^2) = 0$.

Implying that $\mathcal{B}^{*2}\mathcal{B}^2 = 1$, or $\mathcal{B}^*\mathcal{B} = 1$
Using (2), it follows that, $(\mathcal{B}^* + \mathcal{B})^2 = \mathcal{B}^{*2} + 2\mathcal{B}^2 + \mathcal{B}^2 = 4\mathcal{B}^2$.
Hence, $\mathcal{B}^2$ is a partial isometry as claimed.

Lemma
If an operator $\mathcal{B} \in \mathfrak{B}(H)$ is normal, it is also a θ-operator.

Proof
Assuming $\mathcal{B}$ is normal, then $\mathcal{B} = \mathcal{B}\mathcal{B}^*\mathcal{B}$. From the property of θ-operator, now we have $\mathcal{B}^*\mathcal{B} = \mathcal{B}^*\mathcal{B}(\mathcal{B}^* + \mathcal{B}) = \mathcal{B}^*\mathcal{B}\mathcal{B}^* + \mathcal{B}^*\mathcal{B}\mathcal{B}$ ------------------------------ (3)

Again,

$(\mathcal{B}^* + \mathcal{B})\mathcal{B}^*\mathcal{B} = \mathcal{B}^*\mathcal{B}^*\mathcal{B} + \mathcal{B}\mathcal{B}^*\mathcal{B}$ -------------------------- (4)

From R.H.S of equation(3), we have
$\mathcal{B}^*\mathcal{B}\mathcal{B}^* + \mathcal{B}^*\mathcal{B}\mathcal{B} = \mathcal{B}^*\mathcal{B}^*\mathcal{B} + \mathcal{B}^*\mathcal{B}\mathcal{B}$
$= \mathcal{B}^{*^2}\mathcal{B} + \mathcal{B}^*\mathcal{B}^2$ (since $\mathcal{B}$ and $\mathcal{B}^*$ commute)
$= \mathcal{B}^{*^2}\mathcal{B} + \mathcal{B}\mathcal{B}^*\mathcal{B}$,

Which is similar to the R.H.S of equation (4).
Therefore, every normal operator is a θ-operator.

Theorem
Let $\mathcal{A}, \mathcal{B} \in \mathfrak{B}(H)$. If $\mathcal{A}$ is unitarily equivalent to $\mathcal{B}$, denoted by $\mathcal{A} \cong \mathcal{B}$ and $\mathcal{A}$ is a θ-operator, then so is $\mathcal{B}$.

Proof: Since $\mathcal{A}$ is unitarily equivalent to $\mathcal{B}$, there exist a unitary operator $\mathcal{U}$, such that $\mathcal{A}\mathcal{U} = \mathcal{B}\mathcal{U}$. i.e, $\mathcal{A} = \mathcal{U}^*\mathcal{B}\mathcal{U}$ and $\mathcal{A}^* = \mathcal{U}^*\mathcal{B}^*\mathcal{U}$.
Thus,
$(\mathcal{A}^*\mathcal{A}) = (\mathcal{U}^*\mathcal{B}^*\mathcal{U})(\mathcal{U}^*\mathcal{B}\mathcal{U})$
$= \mathcal{U}^*\mathcal{B}^*\mathcal{U}\mathcal{U}^*\mathcal{B}\mathcal{U}$
$= \mathcal{U}^*\mathcal{B}^*\mathcal{B}U$
$= \mathcal{U}^*\mathcal{B}^*\mathcal{U}\mathcal{B}$
$= \mathcal{U}^*\mathcal{U}(\mathcal{B}^*\mathcal{B})$, But $\mathcal{U}^*\mathcal{U} = I$,

hence,

$(\mathcal{A}^*\mathcal{A}) = \mathcal{B}^*\mathcal{B}$ ------------------ (5)

And

$(\mathcal{A}^* + \mathcal{A}) = \mathcal{U}^*\mathcal{B}^*\mathcal{U} + \mathcal{U}^*\mathcal{B}\mathcal{U}$
$= \mathcal{U}^*\mathcal{U}\mathcal{B}^* + \mathcal{U}^*\mathcal{U}\mathcal{B}$
$= \mathcal{U}^*\mathcal{U}(\mathcal{B}^* + \mathcal{B})$.

Hence,

$\mathcal{A}^* + \mathcal{A} = \mathcal{B}^* + \mathcal{B}$ --------------- (6)

Again, $\mathcal{A}^*\mathcal{A}(\mathcal{A}^* + \mathcal{A}) = \mathcal{A}^*\mathcal{A}\mathcal{A}^* + \mathcal{A}^*\mathcal{A}\mathcal{A}$
$= \mathcal{U}^*\mathcal{B}^*\mathcal{U}\mathcal{U}^*\mathcal{B}\mathcal{U}\mathcal{U}^*\mathcal{B}^*\mathcal{U} + \mathcal{U}^*\mathcal{B}^*\mathcal{U}\mathcal{U}^*\mathcal{B}\mathcal{U}\mathcal{U}^*\mathcal{B}\mathcal{U}$
$= \mathcal{U}^*\mathcal{B}^*\mathcal{B}\mathcal{B}^*\mathcal{U} + \mathcal{U}^*\mathcal{B}^*\mathcal{B}\mathcal{B}\mathcal{U}$
$= \mathcal{U}^*\mathcal{U}\mathcal{B}^*\mathcal{B}\mathcal{B}^* + \mathcal{U}^*\mathcal{U}\mathcal{B}^*\mathcal{B}\mathcal{B}$
$= \mathcal{U}^*\mathcal{U}(\mathcal{B}^* + \mathcal{B})$ ----------------------- (7)
And

$(\mathcal{A}^* + \mathcal{A})\mathcal{A}^*\mathcal{A} = \mathcal{A}^*\mathcal{A}^*\mathcal{A} + \mathcal{A}\mathcal{A}^*\mathcal{A}$
$= \mathcal{U}^*\mathcal{B}^*\mathcal{U}\mathcal{U}^*\mathcal{B}^*\mathcal{U}\mathcal{U}^*\mathcal{B}\mathcal{U} + \mathcal{U}^*\mathcal{B}\mathcal{U}\mathcal{U}^*\mathcal{B}^*\mathcal{U}\mathcal{U}^*\mathcal{B}\mathcal{U}$
$= \mathcal{U}^*\mathcal{B}^*\mathcal{B}^*\mathcal{B}\mathcal{U} + \mathcal{U}^*\mathcal{B}\mathcal{B}^*\mathcal{B}\mathcal{U}$
$= \mathcal{U}^*\mathcal{B}^*\mathcal{U} + \mathcal{U}^*\mathcal{B}\mathcal{U}$
$= \mathcal{U}^*\mathcal{U}(\mathcal{B}^* + \mathcal{B})$ −−−−−−−−−−−−−−−−−−−−−−−−(8)

From (5) and (6) and comparing the R.H.S of equation (7) and the R.H.S of equation (8), they are equal. Hence, $\mathcal{T}$ is also a θ-operator?

Proposition
If two unitary operators $\mathcal{A}, \mathcal{B} \in \mathfrak{B}(H)$ are such that $\mathcal{A} \overset{a.s}{\sim} \mathcal{B}$ and $\mathcal{A}$ is a θ-operator, then $\mathcal{B}$ is also a θ-operator.

Proof
$\mathcal{A} \overset{a.s}{\sim} \mathcal{B}$ Implies that, an invertible operator $\mathcal{N}$ exists such that

$\mathcal{A}^*\mathcal{A} = \mathcal{N}^{-1}\mathcal{B}^*\mathcal{B}\mathcal{N}$ −−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−− (9)

and

$\mathcal{A}^* + \mathcal{A} = \mathcal{N}^{-1}(\mathcal{B}^* + \mathcal{B})\mathcal{N}$ −−−−−−−−−−−−−−−−−−−−−−−−−−−− (10)

From (9), we have
$\mathcal{A} = \mathcal{A}\mathcal{N}^{-1}\mathcal{B}^*\mathcal{B}\mathcal{N}$
$= \mathcal{A}\mathcal{N}^{-1}\mathcal{N}\mathcal{B}^*\mathcal{B}$
$= \mathcal{A}\mathcal{B}^*\mathcal{B}$ and thus, $\mathcal{A}^* = (\mathcal{A}\mathcal{B}^*\mathcal{B})^* = \mathcal{B}^*\mathcal{B}\mathcal{A}^*$.

Applying the property of θ-operator, we have;
$\mathcal{A}^*\mathcal{A} = \mathcal{B}^*\mathcal{B}\mathcal{A}^*\mathcal{A}\mathcal{B}^*\mathcal{B} = \mathcal{B}^*\mathcal{B}\mathcal{B}^*\mathcal{B} = (\mathcal{B}^*\mathcal{B})^2 = \mathcal{B}^*\mathcal{B}$ (Projection property). Also,
$\mathcal{N}^{-1}(\mathcal{B}^* + \mathcal{B})\mathcal{N} = \mathcal{A}^* + \mathcal{A} = \mathcal{B}^*\mathcal{B}\mathcal{A}^* + \mathcal{A}\mathcal{B}^*\mathcal{B}$. But $\mathcal{A} \cong \mathcal{B}$, then it implies there exists a unitary operator, $\mathcal{U}$, such that $\mathcal{A} = \mathcal{U}^*\mathcal{B}\mathcal{U}$ and $\mathcal{A}^* = \mathcal{U}^*\mathcal{B}^*\mathcal{U}$.
Thus, $\mathcal{A}^* + \mathcal{A} = \mathcal{B}^*\mathcal{B}\mathcal{A}^* + \mathcal{A}\mathcal{B}^*\mathcal{B}$
$= \mathcal{B}^*\mathcal{B}\mathcal{U}^*\mathcal{B}^*\mathcal{U} + \mathcal{U}^*\mathcal{B}\mathcal{U}\mathcal{B}^*\mathcal{B}$
$= \mathcal{B}^*\mathcal{B}\mathcal{B}^*\mathcal{U}^*\mathcal{U} + \mathcal{U}^*\mathcal{U}\mathcal{B}\mathcal{B}^*\mathcal{B}$
$= \mathcal{B}^*\mathcal{B}\mathcal{B}^* + \mathcal{B}\mathcal{B}^*\mathcal{B}$
$= \mathcal{B}^*\mathcal{B}(\mathcal{B}^* + \mathcal{B}\mathcal{B})$, but $\mathcal{B}^*\mathcal{B} = I$, thus
$= \mathcal{B}^* + \mathcal{B}$.

This shows that $\mathcal{B}$ is also a θ-operator.

Remark
If $\mathcal{A}, \mathcal{B} \in \mathfrak{B}(H)$ are such that $\mathcal{A} \overset{a.s}{\sim} \mathcal{B}$ and if $\mathcal{A}$ is normal then $\mathcal{B}$ is also normal since normal operators are contained in θ-operators.

Theorem.
Let $\mathcal{A}, \mathcal{B} \in \mathfrak{B}(H)$. If $\mathcal{A} \overset{a.s}{\sim} \mathcal{B}$ and $\mathcal{A}$ is posinormal, then $\mathcal{B}$ is also posinormal.

Proof
Since $\mathcal{A}$ is posinormal, it implies that, $\mathcal{A}\mathcal{A}^* = \mathcal{A}^*P\mathcal{A}$, where P is an interrupter. Also, since $\mathcal{A} \overset{a.s}{\sim} \mathcal{B}$, then there exist an invertible operator $\mathcal{N}$ such that $\mathcal{B}^*\mathcal{B} = \mathcal{N}^{-1}\mathcal{A}^*\mathcal{A}\mathcal{N}$ and $\mathcal{B}^* + \mathcal{B} = \mathcal{N}^{-1}(\mathcal{A}^* + \mathcal{A})\mathcal{N}$.
Assuming $\mathcal{A}$ is an isometry, then from $\mathcal{A}\mathcal{A}^* = \mathcal{A}^*P\mathcal{A}$, we have $\mathcal{A} = \mathcal{A}^*P\mathcal{A}\mathcal{A}$ and therefore, $\mathcal{A}^* = (\mathcal{A}^*P\mathcal{A}\mathcal{A})^* = \mathcal{A}^*\mathcal{A}^*P^*\mathcal{A}$.

Hence,
$\mathcal{B}^*\mathcal{B} = \mathcal{N}^{-1}\mathcal{A}^*\mathcal{A}^*P^*\mathcal{A}\mathcal{A}^*P\mathcal{A}\mathcal{A}\mathcal{N}$
$= \mathcal{N}^{-1}\mathcal{A}^*\mathcal{A}^*P^*P\mathcal{A}\mathcal{A}\mathcal{N}$
$= \mathcal{N}^{-1}\mathcal{A}^*\mathcal{A}^*\mathcal{A}\mathcal{A}\mathcal{N}$
$= \mathcal{N}^{-1}\mathcal{A}^*\mathcal{A}\mathcal{N}$ and
$\mathcal{B}^* + \mathcal{B} = \mathcal{N}^{-1}(\mathcal{A}^*\mathcal{A}^*P^*\mathcal{A} + \mathcal{A}^*P\mathcal{A}\mathcal{A})\mathcal{N}$.
$= \mathcal{N}^{-1}(\mathcal{A}^*\mathcal{A}^*\mathcal{A}P^* + P\mathcal{A}^*\mathcal{A}\mathcal{A})\mathcal{N}$.
$= \mathcal{N}^{-1}\mathcal{A}^*\mathcal{A}(\mathcal{A}^*P^* + P\mathcal{A})\mathcal{N}$.
$= \mathcal{N}^{-1}(\mathcal{A}^*P^* + P\mathcal{A})\mathcal{N}$, but $P \geq 0$,
thus we have
$= \mathcal{N}^{-1}(\mathcal{A}^* + \mathcal{A})\mathcal{N}$,

Since the posinormality of $\mathcal{A}$ justifies the almost similarity property with $\mathcal{B}$ and vice versa, then $\mathcal{B}$ is posinormal. Hence, any posinormal operators which are similar and unitarily equivalent are also almost similar.

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