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On The Norm of Finite Length Elementary Operator in Tensor Product of C*-Algebras

Peter Guchu Muiruri[1], Denis Njue King'ang'i[2], Sammy Musundi Wabomba[3]

[1,3] Department of Physical Sciences, Chuka University, Chuka, Kenya

[2] Department of Mathematics and Computer Science, University of Eldoret,Eldoret, Kenya

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Corresponding Author: **Peter Guchu Muiruri**

ABSTRACT

Many properties of Elementary operators, including spectrum, numerical ranges, compactness, rank, and norm have been studied in depth and some results have been obtained. However, little has been done in determining the norm of finite length elementary operator in tensor product of C*-algebras. The norm of basic elementary operator in tensor product of C*-algebras have been determined and results obtained. This paper determines the norm of finite length elementary operator in a tensor product of C*-algebras. More precisely, the bounds of the norm of finite length elementary operator in a tensor product of C*-algebras are investigated. The paper employs the techniques of tensor products and finite rank operators to express the norm of an elementary operator in terms of its coefficient operators.

KEYWORDS: Finite Length Elementary Operator, Finite Rank Operator, Tensor Product and C*-algebras

I. INTRODUCTION

Let $H \otimes K$ be tensor product of complex Hilbert spaces H and K, and $B(H \otimes K)$ be the set of bounded linear operators on $H \otimes K$. Let $A \otimes B, C \otimes D$ being fixed elements of $B(H \otimes K)$, where $A, C \in B(H)$, the set of bounded linear operators on H and $B, D \in B(K)$, the set of bounded linear operators on K. Then we have the following definitions:-

An elementary operator, $T_n : B(H \otimes K) \to B(H \otimes K)$, is defined as;

$$T_n(X \otimes Y) = \sum_{i=1}^{n} A_i \otimes B_i(X \otimes Y)C_i \otimes D_i, \forall X \otimes Y \in B(H \otimes K)$$

When n = 1 we obtain the basic elementary operator, $M_{H \otimes K} : B(H \otimes K) \to B(H \otimes K)$, defined as;

$$M_{H \otimes K}(X \otimes Y) = A \otimes B(X \otimes Y)C \otimes D, \forall X \otimes Y \in B(H \otimes K).$$

When n = 2 then we obtain an elementary operator of length two which is defined by;

$$T_2(X \otimes Y) = A_1 \otimes B_1(X \otimes Y)C_1 \otimes D_1 + A_2 \otimes B_2(X \otimes Y)C_2 \otimes D_2,$$

$\forall X \otimes Y \in B(H \otimes K)$.

The Jordan elementary operator, $U_{H \otimes K}$: B(H ⊗K) → B(H ⊗K), is defined as;

$$U_{H \otimes K}(X \otimes Y) = A \otimes B(X \otimes Y)C \otimes D + C \otimes D(X \otimes Y)A \otimes B, \forall X \otimes Y \in B(H \otimes K).$$

This paper discusses the norm of elementary operator. In section II it reviews the norm of elementary operator in general C*-algebra and in section III it looks at the norm of elementary operator in tensor product of C*-algebras before embarking on the main result on the norm of finite length elementary operator in tensor product of C*-algebras in section IV.

II. Norm of elementary operator in general C*-algebra

Previous studies have shown the determination of norm of elementary operator in C*-algebras, JB*-algebra, standard operator algebra, cartan factor, prime C*-algebra, two-dimensional complex Hilbert space and tensor product. Different researchers have been attracted to the study of the norm of elementary operator in C*-algebra due to the wide range of applications of C*-algebras. Timoney (2001) determined the norm of basic elementary operator in C*-algebra and obtained the following result;

Theorem 2.1 :(Timoney, 2001)

Let ϑ be C*-algebra, then $\| M_{A,B} \| = sup\{\| M_{A,B}(U) \| : U \in U(\vartheta)\} = sup\{\| AUB \| : U \in U(\vartheta)\}$
where $U(\vartheta)$ denotes the set of unitaries in ϑ.

Timoney (2001) created the basis for determining the upper bound of the norm of finite length elementary operator in tensor product of C*-algebra.

Okelo and Agure (2011) used the finite rank operators to determine the norm of the basic elementary operator in C*-algebra and proved Lemma 2.2 below.

Lemma 2.2: (Okelo and Agure, 2011).
Let H be a Hilbert space, B(H) the algebra of bounded linear operators on H. If $M_{A,B} : B(H) \to B(H)$ is defined by $M_{A,B}(X) = AXB, \forall X \in B(H)$ where A and B are fixed in $B(H)$ then $\| M_{A,B}(X) \| = \| A \| \| B \|$, with $\| X \| = 1$ where $X(x) = x, \forall$ unit vectors $x \in H$.

King'ang'i et al. (2014) extended the work of Okelo and Agure (2011) and determined the norm of elementary operator of length two for finite-dimensional separable Hilbert space $W \in B(H)$ with $\| W \| = 1$ and $W(x) = x$ for all unit vectors $x \in H$ and proved theorem 2.3;

Theorem 2.3: (King'ang'i et al., 2014).
Let H be a complex Hilbert space and $B(H)$ be algebra of all bounded linear operators on H. Let E_2 be the elementary operator on $B(H)$. If for an operator $W \in B(H)$ with $\| W \| = 1$, we have $W(x) = x$ with all unit vectors $x \in H$, then $\| E_2 \| = \sum_{i=1}^{2} \| A_i \| \| B_i \|$.

King'ang'i (2017) employed the concept of the maximal numerical range of A*B relative to S to determine the lower bound of the norm of an elementary operator of length two and solved theorem 2.4 ;

Theorem 2.4: (King'ang'i, 2017).
Let E_2 be an elementary operator of length two on $B(H)$. Then

$$\| E_2 \| \geq Sup_{\lambda \in W_B(A^*B)} \| \| B_1 \| A_1 + \frac{\bar{\lambda}}{\| B_1 \|} A_2 \|.$$

King'ang'i (2017) also determined the conditions under which the norm of an elementary operator of length two is expressible in terms of the norms of its coefficients operators by proving Corollary 2.5;

Corollary 2.5: (King'ang'i, 2017).
Let H be a complex Hilbert space and A_i, B_i be bounded linear operators on H for $i = 1, 2$. Let $0 \in W_{B_1}(B_1^* B_2) \cup W_{B_2}(B_1^* B_2)$. Then $\| E_2 \| \geq \| A_1 \| \| B_1 \|$, where E_2 is the elementary operator of length two.

Theorem 2.6: (King'ang'i, 2017).
Let H be a complex Hilbert space and A_i, B_i be bounded linear operators on H for i = 1, 2. Let E_2 be an elementary operator of length two. *If* $\| A_1 \| \| A_2 \| \in W_{A_1}(A_2 A_1^*)$ and $\| B_1 \| \| B_2 \| \in W_{B_2}(B_1^* B_2)$ then

$$\| E_2 \| = \sum_{i=1}^{2} \| A_i \| \| B_i \|.$$

Kawira et al. (2018) extended the work of King'ang'i et al. (2014) to finite length and determined the norm of an elementary operator of an arbitrary length in a C*-algebra using finite rank operators and proved theorem 2.7;

Theorem 2.7: (Kawira et al., 2018).
Let H be complex Hilbert spaces and $B(H)$ be the algebra of bounded linear operators on H. Let E_n be elementary operator on $B(H)$.If $\forall X \in B(H)$ with $\| X \| = 1$, we have $X(f) = f$ for all unit vector f ∈ H then $\| E_n \| = \sum_{i=1}^{n} \| A_i \| \| B_i \|, n \in \mathbb{N}$.

Kawira et al. (2018) is very important for the methodology of determining the norm of finite length elementary operator in tensor product of C*-algebras.

III. Norm of elementary operator in tensor product of C*-algebras

Muiruri et al. (2018) determined the norm of basic elementary operator in a tensor product of C*-algebras using the finite rank operator and properties of tensor product and proved the following theorem 3.1;

Theorem 3.1: (Muiruri et al., 2018).
Let H and K be complex Hilbert spaces and $B(H \otimes K)$ be the set of bounded linear operators on $H \otimes K$. Then $\forall X \otimes Y \in B(H \otimes K)$ with $\| X \otimes Y \| = 1$, we have $\| M_{A \otimes B, C \otimes D} \| = \| A \| \| B \| \| C \| \| D \|$, where A, C and B, D are fixed elements in $B(H)$ and $B(K)$ respectively.

As a consequence of the above Muiruri et al. (2018) related the norm of basic elementary operator in tensor product and the usual norm in this elementary operator in C*-algebra and arrived at the corollary 3.2 below;

Corollary 3.2: (Muiruri et al., 2018).
Let H and K be complex Hilbert spaces and $B(H \otimes K)$ be the set of bounded linear operators on $H \otimes K$. Then
Then
$\forall X \otimes Y \in B(H \otimes K)$ with $\| X \otimes Y \| = 1$, we have $\| M_{A \otimes B, C \otimes D} \| = \| M_{A,C} \| \| M_{B,D} \|$, where $M_{A,C}$ and $M_{B,D}$ are basic elementary operators in $B(H)$ and $B(K)$ respectively.

Then Daniel et al. (2022) used the stampli's maximal numerical range to determine the norm of basic elementary operator in a tensor product and they obtained the following theorem 3.3;

Theorem 3.3: (Daniel et al., 2022).

Let H and K be Hilbert spaces and let $M_{A\otimes B,C\otimes D}$ be basic elementary operator on $B(H\otimes K)$ the set of complex Hilbert space $H\otimes K$.If $\forall U\otimes V\in B(H\otimes K)$ with $\| U\otimes V\| = 1$, $A,C\in B(H),B,D\in B(K)$ $\zeta\in W_0(C),\xi\in W_0(D)$ then we have
$\| M_{A\otimes B,C\otimes D}\backslash B(H\otimes K)\| =$
$Sup_{\zeta\in W_0(C)}Sup_{\xi\in W_0(D)}\{|\zeta||\xi|\,\| A\|\| B\|\}$

Finally, Daniel et al., (2023) determined the bounds of the norm of elementary operator of length two in tensor product using the Stampli's maximal numerical range and obtained theorem 3.4;

Theorem 3.4: (Daniel et al., (2023).
Let H and K be Hilbert spaces and let $M_{2A\otimes B,C\otimes D}$ be basic elementary operator on $B(H\otimes K)$ the set of complex Hilbert space $H\otimes K$.If $\forall U\otimes V\in B(H\otimes K)$ with $\| U\otimes V\| = 1$, $A_i,C_i\in B(H),B_i,D_i\in B(K)$ $\zeta_i\in W_0(C_i),\xi_i\in W_0(D_i)$ then we have
$\| M_{2A\otimes B,C\otimes D}\backslash B(H\otimes K)\| =$
$Sup_{\zeta_i\in W_0(C_i)}Sup_{\xi_i\in W_0(D_i)}\{|\zeta_i||\xi_i|\,\| A_i\|\| B_i\|\}$

IV. NORM OF FINITE LENGTH ELEMENTARY OPERATOR IN TENSOR PRODUCT OF C*-ALGEBRAS

In this section, as our main result, we investigate the bounds of the norm of an elementary operator of finite length in a tensor product of C∗-algebras using the concept of finite rank operator and properties of tensor product of C*-algebras

Theorem 4.1
If H and K are complex Hilbert spaces and$B(H\otimes K)$, the set of bounded linear operator on $H\otimes K$. If $\forall X\otimes Y\in B(H\otimes K)$ and $\| X\otimes Y\| = 1$ then;

$$\| T_n\| = \sum_{i=1}^{n}\| A_i\|\| B_i\|\| C_i\|\| D_i\|$$

, where T_n is the Elementary operator of finite length as defined earlier and $A_i,C_i\in B(H)$ and $B_i,D_i\in B(K)$.

Proof

By definition
$\| T_n\backslash B(H\otimes K)\| = \sup\{\| T_n(X\otimes Y)\| : \forall X\otimes Y\in B(H\otimes K),\| X\otimes Y\| = 1\}$.

Therefore we have;
$\| T_n\backslash B(H\otimes K)\| \geq \| T_n(X\otimes Y)\|,\quad \forall X\otimes Y\in B(H\otimes K),\| X\otimes Y\| = 1$

Thus, $\forall\varepsilon\geq 0$
$\| T_n\backslash B(H\otimes K)\| -\varepsilon < \| T_n(X\otimes Y)\| \quad \forall X\otimes Y\in B(H\otimes K),\| X\otimes Y\| = 1$

$$\| T_n\backslash B(H\otimes K)\| -\varepsilon < \| \sum_{i=1}^{n} A_i\otimes B_i(X\otimes Y)C_i\otimes D_i\| =$$

$\| A_1\otimes B_1(X\otimes Y)C_1\otimes D_1 + A_2\otimes B_2(X\otimes Y)C_2\otimes D_2 + \cdots + A_n\otimes B_n(X\otimes Y)C_n\otimes D_n\|$

From properties of tensor product of operators we have $A_i\otimes B_i(X\otimes Y) = A_iX\otimes B_iY$

Therefore we have;
$\| T_n\backslash B(H\otimes K)\| -\varepsilon < \| A_1XC_1\otimes B_1YD_1 + A_2XC_2\otimes B_2YD_2 + \cdots + A_nXC_n\otimes B_nYD_n\|$

Therefore, by triangular inequality, we have;
$\| T_n\backslash B(H\otimes K)\| -\varepsilon < \| A_1XC_1\otimes B_1YD_1\| + \| A_2XC_2\otimes B_2YD_2\| + \cdots + \| A_nXC_n\otimes B_nYD_n\|$

Also using the tensor product property that $\| A_iX\otimes B_iY\| = \| A_iX\|\| B_iY\|$

Thus we have;
$\| T_n\backslash B(H\otimes K)\| -\varepsilon < \| A_1XC_1\|\| B_1YD_1\| + \| A_2XC_2\|\| B_2YD_2\| + \cdots + \| A_nXC_n\|\| B_nYD_n\|$

Since $\varepsilon\geq 0$ was arbitrarily taken then

$$\| T_n\backslash B(H\otimes K)\| \leq \| A_1XC_1\|\| B_1YD_1\| + \| A_2XC_2\|\| B_2YD_2\| + \cdots + \| A_nXC_n\|\| B_nYD_n\| \quad (4.1)$$

Since: $\| A_iXC_i\| \leq \| A_i\|\| X\|\| C_i\| = \| A_i\|\| C_i\|$ since $\| X\| = 1$

Thus $\| A_iXC_i\| \leq \| A_i\|\| C_i\|$

Likewise;
$\| B_iYD_i\| \leq \| B_i\|\| D_i\|$

Apply above to equation (4.1) we have
$\| T_n\| \leq \sum_{i=1}^{n}\| A_i\|\| B_i\|\| C_i\|\| D_i\|$

(4.2)

Conversely, let $(e\otimes f)$ be unit vector in $H\otimes K$ where $e\in H$ and $f\in K$ then
$\| T_n(X\otimes Y)(e\otimes f)\| \leq \| T_n(X\otimes Y)\|\| (e\otimes f)\|$
$\leq \| T_n\|\| (X\otimes Y)\|\| (e\otimes f)\| = \| T_n\|\| X\|\| Y\|\| e\|\| f\|$

This implies that: $\| T_n\| \geq \| T_n(X\otimes Y)(e\otimes f)\| =$
$\| \{A_1\otimes B_1(X\otimes Y)C_1\otimes D_1 + A_2\otimes B_2(X\otimes Y)C_2\otimes D_2 + \cdots + A_n\otimes B_n(X\otimes Y)C_n\otimes D_n\}(e\otimes f)\|$
$= \| A_1\otimes B_1(X\otimes Y)C_1\otimes D_1(e\otimes f) + A_2\otimes B_2(X\otimes Y)C_2\otimes D_2(e\otimes f) + \cdots + A_n\otimes B_n(X\otimes Y)C_n\otimes D_n(e\otimes f)\|$

$= \| A_1XC_1e\otimes B_1YD_1f + A_2XC_2e\otimes B_2YD_2f + \cdots + A_nXC_ne\otimes B_nYD_nf\|$
$\| T_n\| \geq$
$\| A_1XC_1e\otimes B_1YD_1f + A_2XC_2e\otimes B_2YD_2f + \cdots + A_nXC_ne\otimes B_nYD_nf\|$

Thus if we square both sides:

$\| T_n\|^2 \geq$
$\| A_1XC_1e\otimes B_1YD_1f + A_2XC_2e\otimes B_2YD_2f + \cdots + A_nXC_ne\otimes B_nYD_nf\|^2$

$= \langle A_1XC_1e\otimes B_1YD_1f + A_2XC_2e\otimes B_2YD_2f + \cdots + A_nXC_ne\otimes B_nYD_nf, A_1XC_1e\otimes B_1YD_1f + A_2XC_2e\otimes B_2YD_2f + \cdots + A_nXC_ne\otimes B_nYD_nf\rangle$
$\|T_n\| \geq \langle A_1XC_1e\otimes B_1YD_1f, A_1XC_1e\otimes B_1YD_1f + A_2XC_2e\otimes B_2YD_2f + \cdots + A_nXC_ne\otimes B_nYD_nf\rangle +$
$\langle A_2XC_2e\otimes B_2YD_2f, A_1XC_1e\otimes B_1YD_1f + A_2XC_2e\otimes B_2YD_2f + \cdots + A_nXC_ne\otimes B_nYD_nf\rangle$

$+ \cdots + \langle A_nXC_ne\otimes B_nYD_nf, A_1XC_1e\otimes B_1YD_1f + A_2XC_2e\otimes B_2YD_2f + \cdots + A_nXC_ne\otimes B_nYD_nf\rangle$
$= \langle A_1XC_1e\otimes B_1YD_1f, A_1XC_1e\otimes B_1YD_1f\rangle +$
$\langle A_1XC_1e\otimes B_1YD_1f, A_2XC_2e\otimes B_2YD_2f\rangle + \cdots$
$+ \langle A_1XC_1e\otimes B_1YD_1f, A_nXC_ne\otimes B_nYD_nf\rangle$
$+ \langle A_2XC_2e\otimes B_2YD_2f, A_1XC_1e\otimes B_1YD_1f\rangle$
$+ \langle A_2XC_2e\otimes B_2YD_2f, A_2XC_2e\otimes B_2YD_2f\rangle + \cdots +$
$\langle A_2XC_2e\otimes B_2YD_2f, A_nXC_ne\otimes B_nYD_nf\rangle + \cdots +$
$\langle A_nXC_ne\otimes B_nYD_nf, A_1XC_1e\otimes B_1YD_1f\rangle +$

$\langle A_nXC_ne\otimes B_nYD_nf, A_2XC_2e\otimes B_2YD_2f\rangle + \cdots +$
$\langle A_nXC_ne\otimes B_nYD_nf, A_nXC_ne\otimes B_nYD_nf\rangle$

Since $\langle X_1\otimes Y_1, X_2\otimes Y_2\rangle = \langle X_1, X_2\rangle\langle Y_1, Y_2\rangle$ we have:

$= \langle A_1XC_1e, A_1XC_1e\rangle\langle B_1YD_1f, B_1YD_1f\rangle +$
$\langle A_1XC_1e, A_2XC_2e\rangle\langle B_1YD_1f, B_2YD_2f\rangle + \cdots +$
$\langle A_1XC_1e, A_nXC_ne\rangle\langle B_1YD_1f, B_nYD_nf\rangle + \langle A_2XC_2e, A_1XC_1e\rangle$
$\langle B_2YD_2f, B_1YD_1f\rangle + \langle A_2XC_2e, A_2XC_2e\rangle\langle B_2YD_2f, B_2YD_2f\rangle + \cdots +$
$\langle A_2XC_2e, A_nXC_ne\rangle\langle B_2YD_2f, B_nYD_nf\rangle + \cdots +$
$\langle A_nXC_ne, A_1XC_1e\rangle\langle B_nYD_nf, B_1YD_1f\rangle + \langle A_nXC_ne, A_2XC_2e\rangle\langle B_nYD_nf, B_2YD_2f\rangle$
$+ \cdots + \langle A_nXC_ne, A_nXC_ne\rangle\langle B_nYD_nf, B_nYD_nf\rangle$
$= \| A_1XC_1e \|^2 \| B_1YD_1f \|^2 + \langle A_1XC_1e, A_1XC_1e\rangle\langle B_1YD_1f, B_2YD_2f\rangle + \cdots$
$+ \langle A_1XC_1e, A_nXC_ne\rangle\langle B_1YD_1f, B_nYD_nf\rangle$
$+ \langle A_2XC_2e, A_1XC_1e\rangle\langle B_2YD_2f, B_1YD_1f\rangle$
$+ \| A_2XC_2e \|^2 \| B_2YD_2f \|^2 + \cdots + \langle A_2XC_2e, A_nXC_ne\rangle\langle B_2YD_2f, B_nYD_nf\rangle$
$+ \cdots + \langle A_nXC_ne, A_1XC_1e\rangle\langle B_nYD_nf, B_1YD_1f\rangle + \langle A_nXC_ne, A_2XC_2e\rangle$
$\langle B_nYD_nf, B_2YD_2f\rangle + + \cdots + \|A_nXC_ne\|^2 \|B_nYD_nf\|^2$ (4.3)

Now, let $u_i, v_i : H \longrightarrow \mathbb{R}^+$ be functionals for $i = 1,2,\dots,n$

Choose vectors $y, z \in H$ and define finite rank operators $A_i = u_i\otimes y$ and $C_i = v_i\otimes z$ on H for $i = 1,2,\dots,n$ by

$A_ie = (u_i\otimes y)e = u_i(e)y \quad \forall\, e \in H$ with $\| e \| = 1$ $i = 1,2,\dots,n$ and $C_ie = (v_i\otimes z)e = v_i(e)z$ with $\| e \| = 1$ $i = 1,2,\dots,n$

Observe that the norm of A_i for $i = 1,2,\dots,n$ is:

$\| A_i \| = \sup\{\| (u_i\otimes y)\, e \| : e \in H, \| e \| \leq 1\}$

$= \sup\{\| u_i(e)y \| : e \in H, \| e \| \leq 1\}$

$= \sup\{|u_i(e)|\, \| y \| : e \in H, \| e \| \leq 1\}$

$= \sup\{|u_i(e)| : e \in H, \| e \| \leq 1\} = |u_i(e)|$

That is $\| A_i \| = |u_i(e)|\ \forall\, e \in H$ with $\| e \| = 1$ $i = 1,2,\dots,n$

Likewise, the norm of C_i is $\| C_i \| = |v_i(e)|\ \forall\, e \in H$ with $\| e \| = 1$ $i = 1,2,\dots,n$

From 3.2 above then:

$\| A_1XC_1e \|^2 = \| (u_1\otimes y)X(v_1\otimes z)e \|^2$

$= \| (u_1\otimes y)Xv_1(e)z \|^2$

$= \| v_1(e)(u_1\otimes y)X(z) \|^2$

$= |v_1(e)|^2 \| (u_1\otimes y)X(z) \|^2$

$= |v_1(e)|^2 \| u_1(X(z))y \|^2$

$= |v_1(e)|^2 |u_1(X(z))|^2 \| y \|^2$

$= |v_1(e)|^2 |u_1(X(z))|^2 = \| A_1 \|^2 \| C_1 \|^2$

Thus

$\| A_1XC_1e \|^2 = \| A_1 \|^2 \| C_1 \|^2$ (4.4)

Thus using the same concept also:

$\| B_1YD_1f \|^2 = \| B_1 \|^2 \| D_1 \|^2$ (4.5)

$\| A_2XC_2e \|^2 = \| A_2 \|^2 \| C_2 \|^2$ (4.6)

$\| B_2YD_2f \|^2 = \| B_2 \|^2 \| D_2 \|^2$ (4.7)

$\| A_nXC_ne \|^2 = \| A_n \|^2 \| C_n \|^2$ (4.8)

$\| B_nYD_nf \|^2 = \| B_n \|^2 \| D_n \|^2$ (4.9)

Also:

$\langle A_1XC_1e, A_2XC_2e\rangle =$
$\langle (u_1\otimes y)X(v_1\otimes z)\, e, (u_2\otimes y)X(v_2\otimes z)\, e\rangle$
$= \langle (u_1\otimes y)Xv_1(e)z, (u_2\otimes y)Xv_2(e)z\rangle$

$= \langle v_1(e)(u_1\otimes y)Xz, v_2(e)(u_2\otimes y)Xz\rangle$

$= \langle v_1(e)u_1(X(z))y, v_2(e)u_2(X(z))y\rangle$

$= v_1(e)u_1(X(z))v_2(e)u_2(X(z))\langle y, y\rangle$
$= v_1(e)u_1(X(z))v_2(e)u_2(X(z))$

Since $v_1(e), u_1(X(z)), v_2(e)$ and $u_2(X(z))$ are all positive real numbers, we have:

$v_1(e) = |v_1(e)| = \| C_1 \|$, $u_1(X(z)) = |u_1(X(z))| = \| A_1 \|$, $v_2(e) = |v_2(e)| = \| C_2 \|$ and $u_2(X(z)) = |u_2(X(z))| = \| A_2 \|$

Thus we have $\langle A_1XC_1e, A_2XC_2e\rangle = v_1(e)u_1(X(z))v_2(e)u_2(X(z)) = \| C_1 \|\| A_1 \|\| C_2 \|\| A_2 \|$

Since the norms of A_i and C_i for $i = 1,2,\dots,n$ are scalars then :

$\langle A_1XC_1e, A_2XC_2e\rangle = \| A_1 \|\| A_2 \|\| C_1 \|\| C_2$ (4.10)

Hence using the same concept as above then:

$\langle B_1YD_1f, B_2YD_2f\rangle = \| B_1 \|\| B_2 \|\| D_1 \|\| D_2 \|$ (4.11)

It then follows that $\langle A_2XC_2e, A_1XC_1e\rangle =$
$\langle (u_2\otimes y)X(v_2\otimes z)e, (u_1\otimes y)X(v_1\otimes z)\, e\rangle$

$= \langle (u_2\otimes y)Xv_2(e)z, (u_1\otimes y)Xv_1(e)z\rangle$

$= \langle v_2(e)(u_2\otimes y)Xz, v_1(e)(u_1\otimes y)Xz\rangle$

$= \langle v_2(e)u_2(X(z))y, v_1(e)u_1(X(z))y\rangle$

$= v_2(e)u_2(X(z))v_1(e)u_1(X(z))\langle y, y\rangle$

$= v_2(e)u_2(X(z))v_1(e)u_1(X(z)) = \| C_2 \|\| A_2 \|\| C_1 \|\| A_1 \|$

Since the norms of A_i and C_i for $i = 1,2,\dots,n$ are scalars then :

$$\langle A_2XC_2e, A_1XC_1e\rangle = \| A_1 \|\| A_2 \|\| C_1 \|\| C_2 \| \qquad (4.12)$$

Thus using the same concept then:

$$\langle B_2YD_2f, B_1YD_1f\rangle = \| B_1 \|\| B_2 \|\| D_1 \|\| D_2 \| \qquad (4.13)$$

$$\langle B_1YD_1f, B_nYD_nf\rangle = \| B_1 \|\| B_n \|\| D_1 \|\| D_n \| \qquad (4.14)$$

$$\langle B_2YD_2f, B_nYD_nf\rangle = \| B_2 \|\| B_n \|\| D_2 \|\| D_n \| \qquad (4.15)$$

$$\langle B_nYD_nf, B_1YD_1f\rangle = \| B_1 \|\| B_n \|\| D_1 \|\| D_n \| \qquad (4.16)$$

$$\langle B_nYD_nf, B_1YD_1f\rangle = \| B_1 \|\| B_n \|\| D_1 \|\| D_n \| \qquad (4.17)$$

$$\langle A_1XC_1e, A_nXC_ne\rangle = \| A_1 \|\| A_n \|\| C_1 \|\| C_n \| \qquad (4.18)$$

$$\langle A_2XC_2e, A_nXC_ne\rangle = \| A_2 \|\| A_n \|\| C_2 \|\| C_n \| \qquad (4.19)$$

$$\langle A_nXC_ne, A_1XC_1e\rangle = \| A_1 \|\| A_n \|\| C_1 \|\| C_n \| \qquad (4.20)$$

$$\langle A_nXC_ne, A_2XC_2e\rangle = \| A_2 \|\| A_n \|\| C_2 \|\| C_n \| \qquad (4.21)$$

Thus substituting equations (4.4) to (4.21) in (4.3) then

$$\| T_n \|^2 \geq \| A_1 \|^2\| B_1 \|^2\| C_1 \|^2\| D_1 \|^2 +$$
$$\| A_1 \|\| B_1 \|\| C_1 \|\| D_1 \|\| A_2 \|\| B_2 \|\| C_2 \|\| D_2 \|$$
$$+ \cdots + \| A_1 \|\| B_1 \|\| C_1 \|\| D_1 \|\| A_n \|\| B_n \|\| C_n \|\| D_n \| +$$
$$\| A_1 \|\| B_1 \|\| C_1 \|\| D_1 \|\| A_2 \|\| B_2 \|\| C_2 \|\| D_2 \|$$
$$+ \| A_2 \|^2\| B_2 \|^2\| C_2 \|^2\| D_2 \|^2 + \cdots +$$
$$\| A_2 \|\| B_2 \|\| C_2 \|\| D_2 \|\| A_n \|\| B_n \|\| C_n \|\| D_n \| + \cdots +$$
$$\| A_1 \|\| B_1 \|\| C_1 \|\| D_1 \|\| A_n \|\| B_n \|\| C_n \|\| D_n \| +$$
$$\| A_2 \|\| B_2 \|\| C_2 \|\| D_2 \|\| A_n \|\| B_n \|\| C_n \|\| D_n \|$$
$$+ \cdots + \| A_n \|^2\| B_n \|^2\| C_n \|^2\| D_n \|^2$$

This implies that:

$$\| T_2 \|^2 \geq \{A_1 \|\| B_1 \|\| C_1 \|\| D_1 \| + \| A_2 \|\| B_2 \|\| C_2 \|\| D_2 \|$$
$$+ \cdots + \| A_n \|\| B_n \|\| C_n \|\| D_n \|\}^2$$

Thus obtaining square root in both sides:

$$\| T_2 \| \geq \| A_1 \|\| B_1 \|\| C_1 \|\| D_1 \| + \| A_2 \|\| B_2 \|\| C_2 \|\|$$
$$D_2 \| + \cdots + \| A_n \|\| B_n \|\| C_n \|\| D_n \|$$

Finally, it's clear that:

$$\| T_n \| \geq \sum_{i=1}^{n} \| A_i \|\| B_i \|\| C_i \|\| D_i \| \qquad (4.22)$$

From (4.2) and (4.22) then

$$\| T_n \backslash B(H \otimes K) \| = \sum_{i=1}^{n} \| A_i \|\| B_i \|\| C_i \|\| D_i \|$$

RECOMMENDATIONS

From the above main result an attempt can be made in solving the norm of finite length elementary operator in tensor product of C*-algebras using a different methodology of Stampli's maximal numerical range.

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