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Mathematical modeling of flow of fertilizer-water mixture through soil and its effect on concentration and plant growth

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Abstract
In this research we aim to demonstrate and explore how mathematical modeling can be used to aid people gain knowledge and understanding of plants and, in particular, interactions between plants, fertilizers, soil, and water. The primary objective is to convince members of the agricultural, mathematical and biological culture of the need to work together in establishing mechanistic simulations with quantitative data to help them understand flow of soil solutions and their effect on plant growth. The mathematical models used are based on the fundamental knowledge of fluid flows and plant growth that is necessitated by nutrient absorption from the soil by plant roots. Such models allow for a deeper understanding of plant science at the most basic level and can aid us in dealing with real-world issues such as food scarcity, soil pollution and global warming in developing countries, Kenya as a target. We used mathematical model equations which have been made to describe fertilizer (contaminant) as well as soil water flow, their concentrations, uptake by a plant root system and plant growth then solve the equations by finite difference and volume methods with the help of MATLAB program. The technique of explicit difference was used to solve the governing equations analytically. The results indicate that as time of simulation increases, the concentration of fertilizer also increases thus increasing the growth factor which in turn affects the length of plant growth.

Keywords: Contaminant transport, finite difference, concentration, plant growth

Introduction
In our research, we create a mathematical model to understand how fertilizer-water mixtures flow in soil and how their rates influence plant growth and development. The knowledge of transport of contaminants through soil in ground water is used in this research together with equations describing plant growth caused by the concentration of inorganic fertilizer around the root surface. The model is used to calculate the amount of fertilizer consumed by plant roots in relation to the plant's actual growth size in length or volume.

In areas with low soil fertility, large quantities of fertilizer are used, whereas in areas with higher soil fertility, small amounts of fertilizer are used to increase soil fertility. The government has no regulations on the amount of fertilizer to be applied hence no control over the hazards that may be caused on plants as a result of excessive use of fertilizer in soil. In recent years, there has been an unpredictable change in seasons of rainfall and sunshine thus pausing danger on food security in our country Kenya. One of the factors that may lead to such changes is due to continuous use of soluble chemicals present in fertilizer which end up affecting plant populations and possibly causing the extinction of certain less adapted species. This research is based on the fact that in Uasin Gishu County, Kenya maize is a major crop grown in the area. Maize crops thrive well in Loamy soil and in less sloppy land having enough Rainfall supply. Our work presents mathematical models that would be useful in monitoring the rate of flow of fertilizer-water mixture and its concentration in soil and on the root, surface therefore help control amount of fertilizer as well as growth of plants.

Governing equations
The momentum equation, which expresses mass conservation (continuity equation), is generally

required for the hydrodynamic definition of a fluid-flow problem and a relationship between states among them: temperature, stress, and density. The problem with the flow is thus described mathematically by a system of partial differential equations, which can be more or less complex, the solution of which necessitates the Boundary conditions and, if the flow is unsteady, initial conditions describing the particular flow situation are specified.
The movement of fluids inside the soil can be studied on a minuscule scale by taking the soil into account as a dispersed system and, even if only conceptually, solving the problem using the Navier-Stokes's equations. The Navier-Stokes's equation written in x Cartesian co-ordinates is;

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad (1.1)$$

Flow velocities in real time have significant magnitude and direction differences due to the complexities of the paths taken by single fluid particles as they pass through the pores that are intertwined in soil, Consequently, a careful examination of the definition of flow pattern at any point in the domain is virtually impossible. The understanding of flux density, on the other hand, is of greater importance in many applications.
As a result, processes of soil movement and transport are usually defined at a macroscopic level by specifying a Representative Elementary Volume (REV) and a collection of averaged values and balance equations.
We first make an assumption that the fertilizer is soluble in water and the solution is a homogeneous mixture. The fertilizer is rich in nutrients required by plants during growth and development. The velocity/speed which the fertilizer solution flows in soil has direct effects on uptake by plant root system hence affecting plant growth.
In our research, the model considers the equation that regulates vertical, unsaturated –convective or isotropic –soil water flow and contaminant transport. We use the following equation for transport of contaminants through soil in groundwater:

$$D_L\frac{\partial^2 C}{\partial x^2} - \mathrm{V}\frac{\partial C}{\partial x} = \mathrm{T}\frac{\partial C}{\partial t} \quad (1.2)$$

x: denotes the vertical space co-ordinate (m)
t: is time in seconds (s)
D_L: longitudinal dispersion coefficient
V: seepage or pore speeds on average (ms^{-1})
T: is retardation factor
C: is contaminant (Fertilizer) concentration Which supplies nutrients to the plant (cm^3).

The hydraulic properties of the studied soil system are used to simulate water flow and solute transport through soil medium. Direct measurements and tests on soil hydraulic properties can be performed in the laboratory or on the ground through experimentation However, since these approaches can be expensive and take a long time, different methods for estimating soil hydraulic properties have arisen. These properties can be estimated using soil knowledge that is readily obtainable and simple to calculate. Functions in mathematics for estimating soil hydraulic variables using simple soil data for example soil texture, the amount of organic matter, and densities of bulk are applied in this research.

Solute transport modeling
Below are a few of the most important processes which influence transport of solutes in soil. Advection is the average rate of solute mass transportation caused by the flow of water.
Advection can be represented by the Darcy's equation as: $Q = KA\frac{\emptyset_1 - \emptyset_2}{\Delta s}$
If only advection regulated solute transport in the soil, the transport velocity will be the same as the average water flux.

Hydrodynamic dispersion
This process involves a solute spreading through mechanical dispersion and diffusion of molecules in the direction of the flow (longitudinal dispersion) and parallel to the ground (perpendicular or transverse dispersion). The soil undergoes mechanical dispersion resulting from variations in pore size, differences in the length of the flow path, continuous mixing between pores (because of the configuration of the soil's pores), and differences in velocity of pore transport. On both a micro (within pores) and macro (outside pores) scales (preferential flow through soil cracks), this happens in soil medium. Solutes are transferred through molecular diffusion as a result of a concentration gradient.
Dispersion by mechanical means and diffusion of molecules are two main processes that cause hydrodynamic dispersion. Constituents pass from a high-concentration area to a low-concentration area in this process; the greater concentration difference, the faster the diffusion rate.

The equation for advection-dispersion (ADE)
The most widely used mathematical description for transport of solutes through soil is the Advection Dispersion Equation (ADE). The value of the dispersion coefficient is needed for numerical or analytical ADE solutions. Field or laboratory experiments may be used to measure the hydrodynamic dispersion coefficient. Therefore, advection dispersion equation is written mathematically as:

$$\frac{\partial(\theta.C)}{\partial t} = \frac{\partial}{\partial z}\left[D_{sh}(V,\theta)\frac{\partial C}{\partial z}\right] - \frac{\partial(q.c)}{\partial z} - S_s$$

Methodology

In this analysis, we developed a one-dimensional model that can describe water-fertilizer movement, which is used to solve the Richards equation implicitly for unsaturated porous media using the finite element method for volume control, yielding water-fertilizer transport and general expressions for initial and boundary conditions. The rule of conservation of matter governs all water-solute movement in porous media (soil).

Fundamental groundwater flow equations

Darcy law

Henry Darcy, who was the publisher of the findings of his experimental research work in 1956, is credited with establishing the theory of groundwater flow. He discovered that total discharge Q is directly proportional to area of cross-section A, inversely proportional to length Δs, and proportional to the head difference $\emptyset_1 - \emptyset_2$, mathematically written in the equation called Darcy's equation:

$$Q = K A \frac{\emptyset_1 - \emptyset_2}{\Delta s} \tag{3.4}$$

The quantity $\frac{Q}{A}$ is called specific discharge q and K is the hydraulic conductivity.

If $\emptyset_1 - \emptyset_2 = \Delta\emptyset$ and $\Delta s = 0$, the equation is converted into $q = -k\frac{d\emptyset}{ds}$

According to this equation, the specific discharge q, is direct in proportion to the hydraulic gradient k. Darcy's velocity is another name for the particular discharge. The actual flow velocity (the rate of seepage), V is written mathematically as $V = \frac{Q}{n.A} = \frac{q}{n}$ where n denotes the soil porosity and V denotes the seepage velocity, which is invariably greater than q. In reality, the flow almost never has just one dimension. Darcy's law in its broadest sense is applied, with the assumption that the hydraulic conductivity K is constant in all directions: $q_x = -K\frac{\partial\emptyset}{\partial x}$ $q_y = -K\frac{\partial\emptyset}{\partial y}$ $q_z = -K\frac{\partial\emptyset}{\partial z}$

These equations are written as follows for anisotropic materials:

$$q_x = -K_{xx}\frac{\partial\emptyset}{\partial x} - K_{xy}\frac{\partial\emptyset}{\partial y} - K_{xz}\frac{\partial\emptyset}{\partial z}$$

$$q_y = -K_{yx}\frac{\partial\emptyset}{\partial x} - K_{yy}\frac{\partial\emptyset}{\partial y} - K_{yz}\frac{\partial\emptyset}{\partial z}$$

$$q_z = -K_{zx}\frac{\partial\emptyset}{\partial x} - K_{zy}\frac{\partial\emptyset}{\partial y} - K_{zz}\frac{\partial\emptyset}{\partial z}$$

Continuity equation

Darcy's law offers three motion equations for the given four unknowns $(q_x, q_y, q_z, \emptyset)$. The basic physical theory of mass conservation must be fulfilled by the flow phenomenon. When a simple a soil component (block of soil) is filled with water or some other fluid, regardless of the flow pattern, no mass can be gathered or lost. According to the conservation principle, the total sum of the three measured quantities (mass flow) must equal zero.

$$\frac{\partial(\rho q_x)}{\partial x} + \frac{\partial(\rho q_y)}{\partial y} + \frac{\partial(\rho q_z)}{\partial z} = 0$$

When the density is constant because the fluid is incompressible, the equation reduces to: $\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} = 0$ in Cartesian coordinates, which is known as the Equation of Continuity.

Darcy's law is substituted into the equation of continuity, yielding:

$$\frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} + \frac{\partial^2\emptyset}{\partial z^2} = 0 \text{ or } \nabla^2\emptyset = 0 \tag{3.5}$$

Transport of fertilizer contaminants through soil in groundwater

The most common source of pollution under agricultural lands is fertilizers. As a fertilizer is injected into groundwater, it spreads out and travels with it due to advection (caused by groundwater flow), dispersion (caused by mechanical mixing), and molecular diffusion. The mathematical relationships that exist between these processes are as follows:

$$\frac{\partial}{\partial x_i}\left[D_{ij}\frac{\partial C}{\partial x_j}\right] - \frac{\partial}{\partial x_i}(C.V_i) - \frac{C'W'}{n} = T\frac{\partial C}{\partial t} \tag{3.6}$$

$$V_i = -\frac{K_{ij}}{n}.\frac{\partial h}{\partial x_j}$$

$\mathrm{T} = \left[1 + \frac{\rho_b K_d}{n}\right]$

C represents concentration of contaminant, V_i represents Normal pore water velocity (seepage) in the direction x_i, D_{ij} denotes the dispersion coefficient, K_{ij} denotes the hydraulic conductivity, C' is the fertilizer concentration in the sink fluid, W' denotes flow rate in volume as a percentage of the sink, n represents the amount of porosity in impact, h is the hydraulic head, T denotes the factor of retardation and x_i denotes the cartesian co-ordinate. The equation that describes fertilizer transport in groundwater is represented in one dimension below:

$$D_L \frac{\partial^2 C}{\partial x^2} - V\frac{\partial C}{\partial x} = T\frac{\partial C}{\partial t} \qquad (3.7)$$

Where C represents fertilizer concentration, V represents the seepage pore water velocity on average, D_L is coefficient of longitudinal dispersion and T retardation factor.
The above equation can be rearranged and written as:

$\frac{\partial C}{\partial t} + \frac{V}{T}\frac{\partial C}{\partial x} = \frac{D_L}{T}\frac{\partial^2 C}{\partial x^2}$ which is simplified to the equation below;

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = d\frac{\partial^2 C}{\partial x^2},\ 0 < x < L(t), t \geq 0 \text{ where, } u = \frac{V}{T} \text{ and } d = \frac{D_L}{T} \qquad (3.8)$$

Finite difference method (FDM)
FDM are a class of numerical techniques for solving ordinary and partial differential equations by approximating derivatives with finite differences. Both the spatial domain and time interval are discretized or broken into a finite number of steps and the value of the solution at these discrete points approximated by solving algebraic equations containing finite differences and values from nearby points.

Explicit finite difference method
Using forward difference at time t_n and a second order central difference for the space derivative at position x; (Forward Time Centered Space) applied to diffusion problem e.g.

$$\frac{\partial u}{\partial t} = \propto \frac{\partial^2 u}{\partial x^2} \qquad (3.9)$$

The FTCS scheme is given by:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\propto}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

Letting $r = \frac{\propto \Delta t}{\Delta x^2}$ we get: $u_i^{n+1} = u_i^n + r\left(u_{i+1}^n - 2u_i^n - u_{i-1}^n\right)$

Which is stable where $r = \frac{\propto \Delta t}{\Delta x^2} \leq \frac{1}{2}$

The equations are solved analytically by the explicit finite difference method and the results are programmed by MATLAB.

Results and Discussions
The values given in the tables below are not experimental values but values used for simulation.

Table 1: Effect of concentration of fertilizer solution due to flow over a given time interval and diffusion coefficient; d = 0.01

Time	x values					
	0	**0.18**	**0.36**	**0.54**	**0.72**	**0.9**
0	1	0	0	0	0	0
0.2	1	0.0117	0	0	0	0.9877
0.4	1	0.0233	0.0001	0	0.0006	0.9877
0.6	1	0.0348	0.0004	0	0.0012	0.9877
0.8	1	0.0461	0.0008	0	0.0018	0.9877
1	1	0.0572	0.0013	0	0.0024	0.9877
1.2	1	0.0682	0.002	0	0.003	0.9877
1.4	1	0.0791	0.0028	0.0001	0.0035	0.9877
1.6	1	0.0899	0.0037	0.0001	0.0041	0.9877
1.8	1	0.1005	0.0047	0.0001	0.0047	0.9877
2	1	0.111	0.0058	0.0002	0.0052	0.9877

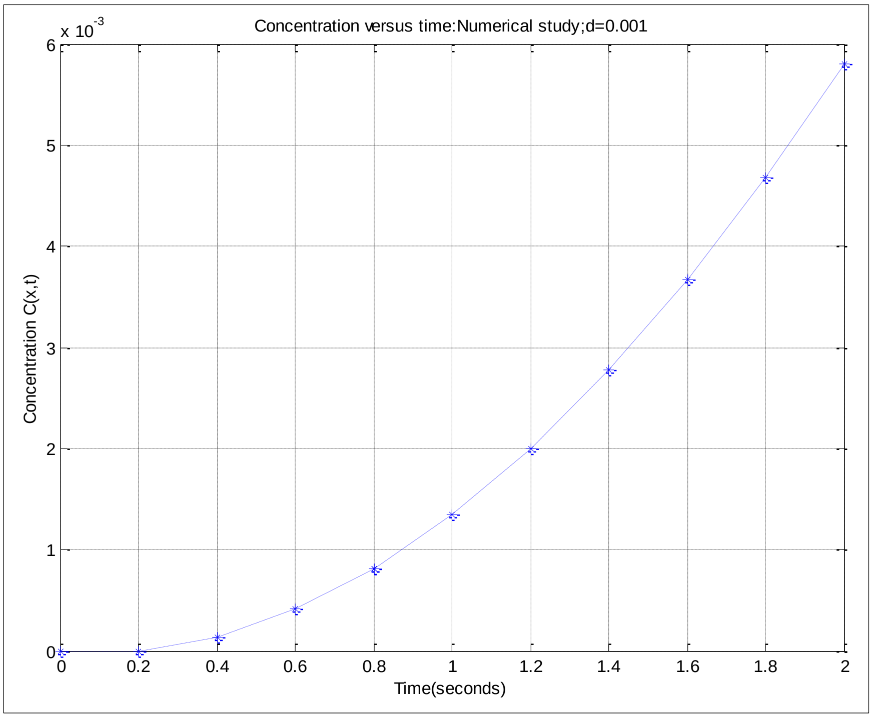

Fig 1: Graph of concentration of fertilizer against time when d = 0.001

Table 2: Effect of concentration of fertilizer solution due to flow over a given time interval and diffusion coefficient; d = 0.0015

Time	x values					
	0	**0.18**	**0.36**	**0.54**	**0.72**	**0.9**
0	1.0000 0	.0000 0	.0000 0	.0000 0	.0000 0	0
0.2	1	0.0148	0	0	0	0.9815
0.4	1	0.0294	0.0002	0	0.0036	0.9816
0.6	1	0.0436	0.0007	0	0.0072	0.9816
0.8	1	0.0576	0.0013	0.0001	0.0107	0.9816
1	1	0.0714	0.0021	0.0001	0.0141	0.9817
1.2	1	0.0849	0.0031	0.0002	0.0175	0.9818
1.4	1	0.0981	0.0043	0.0003	0.0208	0.9819
1.6	1	0.1112	0.0057	0.0004	0.0241	0.9819
1.8	1	0.1239	0.0073	0.0006	0.0273	0.982
2	1	0.1365	0.009	0.0008	0.0304	0.982

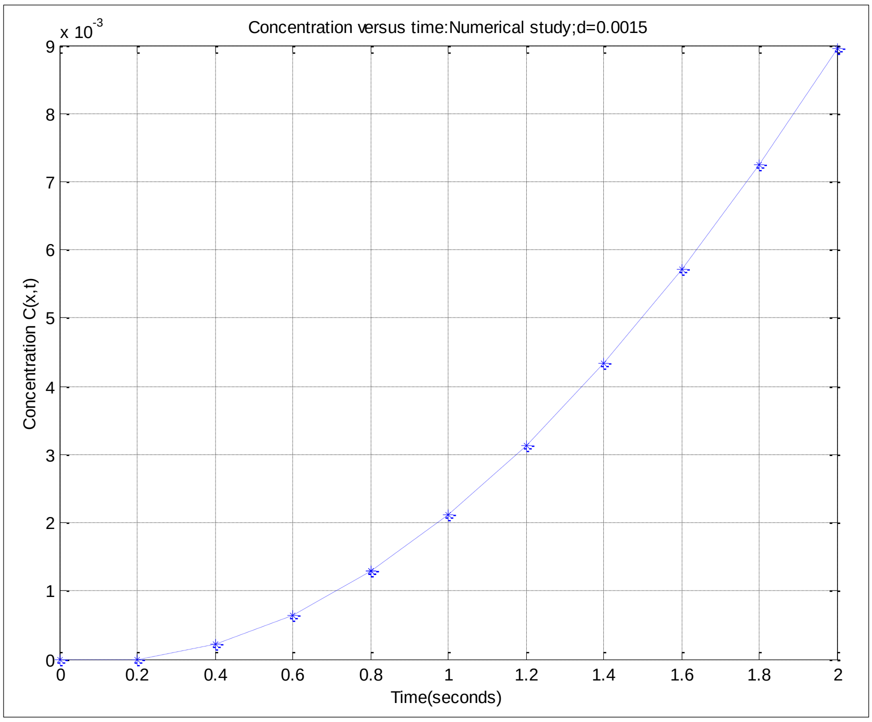

Fig 2: Graph of concentration of fertilizer against time when d = 0.0015

Table 3: Effect of concentration of fertilizer solution due to flow over a given time interval and diffusion coefficient; d = 0.002

Time	x values					
	0	**0.18**	**0.36**	**0.54**	**0.72**	**0.9**
0	1.0000 0	.0000 0	.0000 0	.0000 0	.0000 0	0
0.2	1	0.0179	0	0	0	0.9753
0.4	1	0.0354	0.0003	0	0.0066	0.9755
0.6	1	0.0524	0.0009	0.0001	0.0131	0.9756
0.8	1	0.069	0.0019	0.0002	0.0194	0.9757
1	1	0.0852	0.0031	0.0003	0.0255	0.976
1.2	1	0.101	0.0045	0.0005	0.0315	0.9761
1.4	1	0.1165	0.0062	0.0008	0.0374	0.9763
1.6	1	0.1315	0.0081	0.0012	0.0431	0.9764
1.8	1	0.1462	0.0103	0.0016	0.0487	0.9765
2	1	0.1606	0.0127	0.002	0.0542	0.9767

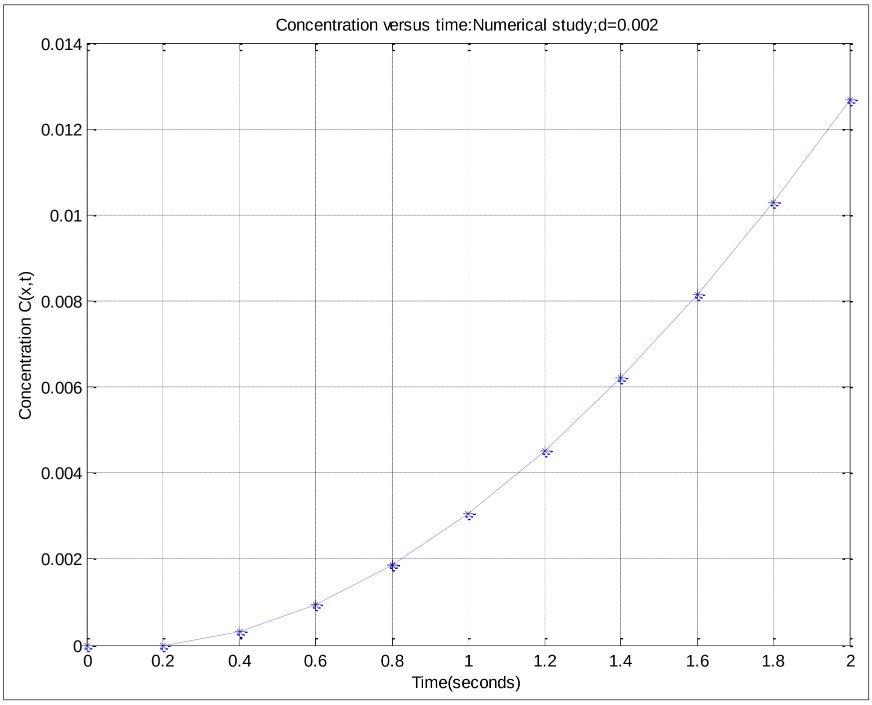

Fig 3: Graph of concentration of fertilizer against time when d = 0.002

Discussions

In this section, results have been presented in tables then analyzed in graphs obtained from output of the MATLAB software.

In fig 1-3, the concentration of fertilizer in soil is constant for some time before rising gradually. This is because the concentration is dependent on the flow through soil and accumulation on the root surface of the plant that takes sometime before it is realized. The results also indicate that an increase in diffusion coefficient causes an increase in concentration of fertilizer in soil over a given time period.

From the results collected and analyzed above, we find that concentration of fertilizer solution, C in soil increases gradually with time, t of flow.

Conclusion

Food shortage in parts of the Kenya has been a problem for years due to environmental factors and poor innovation. With good agricultural practices that apply scientific research to ensure crops thrive well, the problem can be solved. One of the ways is through Mathematical modelling to simulate and explain agricultural processes by formulating and solving equations of fluid flow and crop growth. In this research, equations of solute (fertilizer) transport through soil in underground water was studied and its findings used to model plant growth. The solute concentration taken as fertilizer concentration in soil media over a given time increases causing an increase in growth mitosis factor and finally results in increase in apical (vertical) length of the plant. Such scientific findings play a big role in explaining processes that can improve agricultural practices especially the use of both organic and inorganic fertilizers.

Recommendations

From our research, concentration of fertilizer and the growth mitosis factor coupled with length of growth of plant were emphasized leaving other factors out. In future research, the velocity of flow of fertilizer solution through soil medium can be modelled and studied to see the effect of the speed of flow of the mixture on plant growth. Again, further research can be carried out to check how a specific fertilizer concentration in a given type of soil affects plant/crop growth.

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